Numerical Solution of the Korteweg-de Vries (KdV) Equation

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(Accepted 31 October 1996)

Abstract—A numerical method is developed for solving the Korteweg-de Vries (KdV) equation \( u_t + 6uu_x = 0 \) by using splitting method and quintic spline approximation technique. The convergence, stability and accuracy of the proposed method are discussed. Further, the method is extended to solve the perturbed KdV equation, perturbed by energy influxes of cubic polynomial type, Burger’s type and periodic forcing type, and the effects of these influxes on soliton solution are obtained.

1. INTRODUCTION

The standard Korteweg-de Vries (KdV) equation

\[
    u_t + uu_x + pp_{xxx} = 0,
\]

where \( u \) is a field variable and \( p \) stands for the dispersion parameter, is a nonlinear partial differential equation which models the situation where nonlinearity is of the same order as cubic dispersion. Such situations are rather typical in, for example, hydrodynamics, physics, and acoustics [1-3].

The KdV equation is without any doubt the most celebrated evolution equation and well suited model for conservation of energy process in various wave phenomena. In fact, nature is full of inhomogeneities and in reality the energy is usually unbalanced, so that there exists an outflux or an influx of energy. It is necessary to add more terms to the KdV equation in order to obtain a physical model very close to the nature. The weak energy influx modelled by a right-hand side of the standard KdV equation gives a perturbed KdV equation which models the long waves in a wave guide.

Gardner et al. [4] gave a brief description of how to solve the KdV equation analytically. Among the fundamental works in this field is also the paper by Lax [5] which gives the idea of the Inverse Scattering Technique (IST) to solve the KdV equation. Several researchers have been using Fast Fourier Transform (FFT) technique to simulate the KdV equation numerically. At present there are several excellent monographs available on this topic.

The fecundity of applications which comes across various area of science and technology associated with solids, liquids, gases and plasmas amply justifies numerical analysis as well of the KdV equation. In the present work, an algorithm is developed by using splitting and quintic spline approximation technique to solve the standard KdV equation. A single soliton solution is found to illustrate the accuracy and efficiency of the proposed algorithm. The convergence and stability of the proposed method are discussed. The well-known methods
for integrating the standard KdV equation twice before reaching the final equation solvable analytically do not work in the case of the perturbed KdV equation. Incapabilities of the analytical approaches lead us to numerical methods. The KdV equation perturbed by energy influxes of (a) cubic polynomial type, (b) Burger’s type [6] and (c) periodic forcing type [7] is solved by the method developed, based on operator splitting and quintic spline technique.

2. NUMERICAL METHOD

Consider the KdV equation

\[ \frac{du}{dt} - 6w \frac{du}{dx} + \frac{d^3u}{dx^3} = 0, \quad (1) \]

for all \( t > 0 \) and \(-\infty < x < \infty\), subject to the initial condition

\[ 4G(f(x)) = f(x), \quad (2) \]

where \( F(X) \) is a given function. The function \( f(x) \) must satisfy

\[ \frac{d^n f}{dx^n} \mid_{x=0} = 0, \quad n = 0, 1, 2, 3, \quad (3) \]

in order to guarantee the existence of a unique smooth solution [8].

We split equation (1) in the form

\[ \frac{1}{2 \Delta t} \frac{U_i^n}{\Delta x} = \frac{\partial f(u)}{\partial x}, \quad (4) \]

\[ \frac{1}{2 \Delta t} \frac{\Delta u}{\Delta x} = -\frac{d^3u}{dx^3}, \quad (5) \]

where \( f = 3U^2 \). We approximate the space derivative in equation (4) by the first-order derivative of a quintic spline function interpolating \( f, i = 0, 1, 2, ..., N \), at \( n \) and \( n + 1/2 \) time levels and time derivative by the forward difference to get the approximate form of equation (4) as

\[ U_{i+1}^n + U_{i-1}^n - 2U_i^n = k \theta_i m_i^n + \sqrt{2} + (1 - \theta_i) \kappa, \quad (6) \]

where \( k \) is the increment in time and \( \theta_i, E(0,1) \) is a quintic spline parameter.

The continuity condition [9] for the first-order derivative \( m_i^n \) of quintic spline is

\[ Am_i^n = \Delta \{(\theta_i \kappa + 1) \mathcal{O}_y + \mathcal{O}_x + \mathcal{O}_z \}, \quad (7) \]

where \( \Delta \mathcal{O}_y = W_{i+2} + 26W_{i+1} + 66y + 26W_i + W_{i-1} \) and \( h = \) the space mesh size. Now, eliminating \( m_i^n \) from equations (6) and (7), we obtain the difference scheme for equation (4) as

\[ \Delta U_{n+1}^{\ast} = \frac{120}{24} \frac{\theta_i k}{h} (f_{i+2}^{n+1/2} + 10f_{i+1}^{n+1/2} - 10f_{i-1}^{n+1/2} - f_{i-2}^{n+1/2}) \]

\[ \Delta U_{n}^{\ast} = \frac{120}{24} \frac{(1 - \theta_i) k}{h} (f_{i+2} + 10f_{i+1} - 10f_{i-1} - f_{i-2}). \quad (8) \]

Further, approximating the space derivative in equation (5) by the third-order derivative
of the quintic spline function, and interpolating $U^p_i$, $i = 0, 1, 2, ..., N$, at $n + 1/2$ and $n + 1$ time levels, we get the following form of the equation (5):

$$U^{p+1}_i - U^{n+m} = k e_2^{x+1} - k(1 - e_2)2J^{n+m}, \tag{9}$$

where $e_2 \in (0, 1)$ is a quintic spline parameter.

Now, the continuity condition [9] for the third-order derivative $7I''$ of quintic spline is

$$\frac{120}{2h^2} \left( U^{n+1}_{i+2} - e U^{n+1}_{i+1} - e U^{n+1}_{i-1} - U^{n+1}_{i-2} \right). \tag{10}$$

Eliminating $r$ from equations (9) and (10), we get the difference scheme for equation (5) as

$$\Delta U^{n+1}_i + \frac{120 \theta_k k}{2} \left( U^{n+1}_{i+2} - 2U^{n+1}_{i+1} + 2U^{n+1}_{i-1} - U^{n+1}_{i-2} \right) = Au^{m} + a - e_2^2 \frac{A}{2} \left( V(1/2)^2 - 2f/r_i + 2f/r_i^2 - ft_i \right) \tag{11}$$

For computations, we remove the nonlinearity in the difference scheme by using the Taylor's series expansion

$$f_i^{n+1/2} = f_i^n + \frac{k}{2} \left( \frac{df}{dt} \right)_i^n + O(k^2) = f_i^n + A_i^n (U^{n+1/2} - U^n) + O(k^2), \tag{12}$$

where $A = df/du$.

Finally, equation (8) takes the form

$$\left( 1 - \frac{120 k}{24} \theta_1 A_{n+2} \right) U^{n+1/2}_{i+1} + \sum_{j=1}^{26} 1200 k \theta_1 A_{n+j} U^{n+1/2} = \left( 1 - \frac{120 k}{24} \theta_1 A_{n+2} \right) U^{n+2}_{i+2}$$

$$+ \left( 26 + \frac{120 k}{24} \theta_1 A_{n+1} \right) U^{n+1/2} + \left( 1 + \frac{120 k}{24} \theta_1 A_{n-1} \right) U^{n+1/2} = \left( 1 - \frac{120 k}{24} \theta_1 A_{n+2} \right) U^{n+2}_{i+2}$$

$$+ \left( 26 - \frac{120 k}{24} \theta_1 A_{n+1} \right) U^{n+1} + 66U^n, \quad + \left( 26 + \frac{120 k}{24} \theta_1 A_{n+1} \right) U^{n+1} + \left( 1 + \frac{120 k}{24} \theta_1 A_{n-1} \right) U^{n-1} + \left( 1 + \frac{120 k}{24} \theta_1 A_{n-3} \right)$$

$$\times U^{n-2} + \frac{A}{2} \left( f_i^{n+1} - f_i^n \right), \tag{13}$$

Equations (11) and (13) constitute a multistep implicit finite difference scheme for solving the standard KdV equation (1).

### 3. CONVERGENCE AND STABILITY OF THE METHOD

Local truncation errors in equations (8) and (11) are given by

$$k^2(15(u_n)_n^2 - 60\theta_1 (f_n)_n^2) + k^2 h^2 \left( \frac{15}{4} f_{xxxxx} + 5 \frac{1}{4} g_{xxxx} - 5 \frac{1}{4} g_{xxxxx} + \frac{7}{16} g_{xxxxx} - \frac{7}{4} g_{fxxxxx} \right)$$

$$+ kh^2 \left( \frac{1}{15} u_{xxxxxxx} \right)_{i=1}^{19} \frac{1}{10} g_{fxxxxxxx} + \frac{5}{2} h \left( \frac{1}{15} u_{xxxxxxx} \right)_{i=1}^{19} \frac{1}{10} g_{fxxxxxxx} - 150i(i,j)_n, \tag{14}$$

$$\text{where } u_{xxxxxxx} \text{ and } g_{fxxxxxxx} \text{ are the exact values of the } u \text{ and } g \text{ at } x = x_i, t = t_n.\]
and
\[ k^2(15(u_*)^{n+1/2} + 60\theta_3(u_{xxx})_i^{n+1/2}) + k^3\left(\frac{5}{2}(u_{xx})_i^{n+1/2} + \frac{15}{8}\theta_2(u_{xxxx})_i^{n+1/2}\right) \\
+ k^2h^2\left(\frac{15}{4}(u_{xxx})_i^{n+1/2} + 15\theta_2(u_{xxxxxx})_i^{n+1/2}\right) + kh^4\left(\frac{7}{4}(u_{xxxxxx})_i^m + 3(u_{xxxxxx})_i^j\right) \\
+ k^2h^4\left(\frac{1}{4}(u_{xxxxxxx})_i^m + \epsilon_2(u_{xxxxxxx})_i^m\right). \] (15)

respectively, by neglecting the terms of higher order. The equation (8) gives a truncation error of \(O(k^3 + k^2h^4 + kh^6)\), for \(0, E \in [0, 1]\), and the equation (11) gives truncation error of \(O(k^3 + k^2h^2 + kh^4)\), for \(1_3 \in [0, 1]\), whereas, for \(8, = \frac{1}{2}, \) equation (8) provides a method of \(O(k^2 + h^4)\) and equation (11) provides a method of \(O(k^2 + h^4)\). For discussing the stability and local truncation errors in calculating \(U, F^{*}\) from \(U_0^{*}\), for the proposed algorithm, we eliminate the intermediate value of \(u\) to get the difference relation in \(U^{n+1}\) and \(U^n\) as
\[ \left(\Lambda - \frac{120}{24}\theta_1\frac{k}{h}\Gamma(A^*_i)\right)\left(\Lambda + \frac{120}{2}\theta_2\frac{k}{h^3}\Pi\right)U_i^{n+1} \\
= \left(\Lambda - \frac{120}{24}\theta_1\frac{k}{h}\Gamma(A^*_i)\right)\left(\Lambda + \frac{120}{2}\theta_2\frac{k}{h^3}\Pi\right)U_i^n + \frac{120k}{24}\Gamma(f^n), \] (16)

where \(\text{r} = \text{WY}_+, \text{ low}_-, \text{ WY}_-, \text{ low}_+, \text{ WY}_+, \text{ low}_+, \text{ WY}_-, \text{ WY}_+\).

Further, using the Von Neumann stability method we find that the proposed scheme is stable for \(0, 1/2, i = 1, 2\), if \(0 \leq k/h^3 \leq 0.4\). Further, expanding the term of equation (16) by Taylor series, we find that the truncation error is \(O(k^2 + kh^4)\), for \(8, = 1/2, \). Thus, the proposed scheme is \(O(k^2 + h^4)\), for \(8, = 1/2, \).

4. PERTURBED KdV EQUATION

Nonlinear systems perturbed by energy influx give rise to source-like terms in the governing equation. The weak energy influx modelled by a right-hand side of the KdV equation gives a perturbed KdV equation which models the long waves in a waveguide [6]. We consider the perturbed KdV equation
\[ \frac{du}{dt} - 6u \frac{du}{dx} + \frac{d^3u}{dx^3} = \varepsilon F(\xi), \] (17)

where \(F(u)\) is a smooth function and \(\varepsilon\) is a small parameter.

5. OUTLINES OF THE NUMERICAL METHOD FOR PERTURBED KdV EQUATION

The first step in solving the perturbed KdV equation numerically is to use the method known as operator splitting to remove the inhomogeneous term \&F(U) from equation (17). Thus, we solve the equation
\[ u., - 6uu., + u., = 0, \] (Ito)
which represents the stationary KdV equation. Once equation (18) has been solved then the equation
\[ u_r = \partial F(U) \]  
(19)
is solved. The solution of equation (1) is used to determine the inhomogeneous term \( \partial F(U) \) in equation (19).

6. NUMERICAL RESULTS AND ANALYSIS

In this section, first we consider KdV equation (1) with initial condition
\[ u(x,0) = 2 \text{sech}^2(x). \]
(20)
The single soliton solution of this problem is given by
\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{2} \sqrt{c(x - ct)} \right]. \]
(21)
This solution represents a disturbance that moves in the positive \( x \) -direction at a constant velocity \( c \). For computational work, we choose \( c = 4 \). The results for single soliton solution are plotted in Fig. 1. This shows a comparison of the numerical solution with the soliton solution.

Next, we consider the various functional dependencies of the right-hand side of the perturbed KdV equation (17) modelling various physical phenomena.

Case 2. Let the function \( F(u) \) be a cubic polynomial, i.e.
\[ F(u) = \pm (M - M^2 + bu^3). \]
(22)
We chose \( F(u) = f(U - 1.68 - 0.292) \) and \( E = 0.01 \). Figures 2 and 3 show the change in the amplitude of a solitary wave and the change in the profile of the solution with the

![Fig. 1. Single soliton solution with increasing time.](image)
increase in time. Figure 2 shows an attenuation in the solitary wave with the positive value of $F(u)$ while with negative value of $F(u)$, Fig. 3 shows an amplification.

**Case 2.** Let $F(u)$ model low- and high-frequency 'losses' [10]:

$$\&F(U) = a, u - a, u_{x,x}. \quad (23)$$

1. Let $a_0 = 0$, $a_i > 0$, i.e. only high-frequency losses exist. In fact, this is the case of KdV Burger's equation when the dissipation is weak. Figure 4 is plotted for $a_i = 3$. This clearly shows that after a finite time explosive amplification occurs and the soliton deforms
Numerical solution of the KdV equation

1. Let \( a, c > 0 \), i.e. both low- and high-frequency losses exist. Figure 5 is plotted for \( a_1 = 20, a_1 = 3 \). It is clear from this figure that both low- and high-frequency losses turn to be reversed and amplification occurs.

2. Let \( a, > 0, a_1 > 0 \), i.e. both low- and high-frequency losses exist. Figure 5 is plotted for \( a_1 = 20, a_1 = 3 \). It is clear from this figure that both low- and high-frequency losses turn to be reversed and amplification occurs.

3. Let \( F(u) \) be the periodic sinusoidal type [7]:

\[
\varepsilon F(u) = A \sin(q(x - Vt)),
\]

Fig. 4. Effect of Burger's type (with high-frequency losses) perturbation on soliton.

Fig. 5. Effect of Burger's type (with low- and high-frequency losses) perturbation on soliton.
where $A$ is the perturbation amplitude, $q$ is the wave number defined by $q = (2n/L)n$, $V$ is the perturbation velocity, $L$ is the length of the region of integration and $n$ is an arbitrary integer. The mathematical model based on the forced KdV equation

$$u_t - 6uu_x + u_{xxx} = A \sin(q(x - Vr)) \tag{25}$$

describes a chain of small bodies moving in a shallow or stratified liquid. The equation above is solved numerically with periodic boundary conditions and $L = 256$. For initial condition we take the well-known unperturbed soliton wave

$$u_{\text{soliton}} = \frac{1}{2}c \, \text{sech} \left( \frac{1}{2} \left( \frac{4}{c} (x - Ct) \right) \right),$$

where $c$ is the velocity of the soliton. Following Malomed [7], we have considered only the case where the relative velocity of the soliton $u = V - c$ is much smaller than $V$. Thus, we chose $u = 0.2$, $A = 0.01$, $n = 10$ and $V = 1$ for plotting Fig. 6. This figure shows that the dynamics of the soliton becomes irregular and the soliton demonstrates conspicuous oscillations which are accompanied by a very weak emission of radiation, which produce a small recoil effect on the motion of the soliton.

7. CONCLUSION

Study of Fig. 1 shows that the numerical solution of the standard KdV equation is in good agreement with soliton solution and after a finite time numerical solution perfectly matches with the soliton solution. Thus, the scheme developed yields good computed results for the KdV equation. The numerical method is successfully extended for solving the perturbed KdV equation to analyse the effects of different types of energy influxes on the solitary wave profiles. Thus, we can say that the proposed method may be extended to solve nonlinear problems arising in the theory of solitons.
REFERENCES