Analyticity of semigroup generated by a class of differential operators with interface

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\textit{Keywords:} Interface; Adjoint operator; Analytic semigroup; m-dissipative operators

1. Introduction

Problems involving interface arise naturally in many applied situations, such as acoustic waves in ocean, \cite{2}, thermoelasticity \cite{1}, etc. A systematic study of interface problems involving ordinary differential operators was done in \cite{7} and the references therein.

On the other hand, evolution of a physical system in time is usually written as evolution equation in a Banach space by an initial-value problem for a differential equation of the form:

\begin{equation}
\frac{du}{dt} = Lu(t), \quad t > 0, \quad u(0) = u_0.
\end{equation}

It is important that such problems be well posed in the sense that the solution exists, is unique and depends continuously on the initial data. The problems of type (1) are well posed in a Banach space $X$, if and only if operator $L$ generates a $C_0$ semigroup $S(t)$ in $X$. For an excellent introduction to semigroups of linear operators and application to partial differential equations we refer to \cite{4}. Here solution $u(t)$ is given by $S(t)u_0$ for $u_0 \in D(L)$. An added advantage of the semigroup approach is that it avoids technical complexities and exhibits the dynamic system behaviour of the solution.

In our earlier paper \cite{6}, we have studied a class of differential operators with an interface and shown that such mixed operators generate a $C_0$ semigroup of contractions on an appropriate real Hilbert space $X$. 

The aim of the present work is to establish with suitable assumptions, analyticity of the semigroup generated by differential operators involving an interface in the setting of a complex Hilbert space. The importance of having such an analytic semigroup is in the fact that for \( U(t) = S(t)u_0 \) the solution \( u(t) \) is not only in \( C^1(\mathbb{R}^+; X) \) but is also analytic in time. Besides, we also get that \( u(t) \) is analytic in time. Besides, we also get that \( u(t) \) is analytic in time.

First, we introduce the notations and make a few assumptions. Let \( I_1 = [a, b] \) and \( I_2 = [b, c] \), \(-\infty < a < b < c < \infty\). Let \( L^2(I_2) \) be the Hilbert space of all square integrable complex valued functions on \( I_1 \), endowed with the standard inner product. Our main setting here is the (Cartesian) product Hilbert space \( X = L^2(I_1) \times L^2(I_2) \), endowed with the inner product,

\[
((u_1, u_2), (v_1, v_2)) = (u_1, v_1) + (u_2, v_2), \quad y(u_1, u_2), (v_1, v_2) \in X.
\]

By \( H^2(I_i) \) we mean the following space: For \( i = 1, 2 \),

\[
H^2(I_i) = \{u \in L^2(I_i) / u, u' \text{ exist and are absolutely continuous on } I_i, \text{ and } u'' \in L^2(I_i)\}.
\]

We define \( H = H^2(I_1) \times H^2(I_2) \).

Let \( \{u_1; u_2\} \in H \) and for \( i = 1, 2 \),

\[
\frac{\partial u_i}{\partial t} = P_i(x)u''_i + q_i(x)u'_i + r_i(x)ui
\]

be two linear ordinary differential expressions on \( I_i \).

We make the following assumptions on the coefficients:

(i) The coefficients \( p_i(x), q_i(x), r_i(x) \) are all real valued and \( p_i(x) > 0 \) for all \( x \in I_i, \quad i = 1, 2 \).

(ii) \( p_i \in L^2(I_i), \quad q_i \) is absolutely continuous on \( I_i \), and \( r_i \) is piecewise continuous on \( I_i \), \( i = 1, 2 \).

The corresponding formal Lagrange adjoint expressions are given as

\[
z_i^*u_i = (Piu_i)' - (qUi)' + ru_i, \quad i = 1, 2.
\]

We have the interface conditions at the point \( x = b \) given by,

\[
A[u_1](b) = B[u_2](b),
\]

where \( A \) and \( B \) are \((2 \times 2)\) nonsingular matrices with complex entries and \([u_i](b) = \text{column } (u_i(b), u'_i(b)), \quad i = 1, 2\).

These interface conditions we call matching interface conditions. For a \((2 \times 2)\) matrix \( A \), \( A^* \) denotes the adjoint of \( A \).

In Section 2 we define mixed operator \( L \) and its adjoint \( L^* \), and derive certain identities needed later. In Section 3 we establish the \( m \)-dissipativity of the operator \( L \). Section 4, the main section of paper deals with the analyticity of semigroup generated by \( L \). In Section 5 we give an example to illustrate the applicability of results proved.
2. Mixed operator $L$ and its adjoint

We consider the operator $L$ given by

$$D(L) = \{\{u_x, U_2\} \in H/ A[u_x](b) = B[u_2](b), \quad p_a = p_c = 0\},$$

$$L\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\},$$

(5)

(6)

where

$$P_a = u'(a) - u_1(a)v_a, \quad P_c = u'_2(c) - v_2u_2(c).$$

It can be shown using Theorem 3.6 [8] that $L$ is a densely defined closed unbounded linear operator in $X$ and hence has a unique adjoint.

Also, we have Green's formula given by, for $\{U_1, U_2\} \in D(L)$ and $\{v_x, u_2\} \in H$

$$\int \int (p\bar{v}_x u_x' + q\bar{v}_x u_x)\, dx$$

$$+ \int (p_2v_2u_2' + q_2v_2u_2)\, dx$$

$$+ \sum_{i=1}^{2} \int (\tau_i^* v_i) u_i\, dx$$

= boundary terms $B$ + Interface terms $I$ + Integral terms.

We now simplify the boundary and interface terms.

We introduce the matrices $C_i; \ i = 1, 2$ which allow a simpler representation of the interface conditions satisfied by the adjoint operator and also play a role in establishing the analyticity of the semigroup generated by the operator $L$. For $i = 1, 2$ the matrices $Q$ are given by

$$Q_i = \begin{bmatrix} q_i(b) - p'iib(b) \\ p_i(b) \end{bmatrix}$$

(i) Using the matching conditions $A[u_1](b) = B[u_2](b)$, and the notation

$$K_1 = (A^{-1})^* C_1^*, \quad K_2 = (B^{-1})^* C_2^*,$$

we get

$$\mathcal{F} = \int (\bar{p}(b)v_x(b)u_x(b) + \int (p\bar{v}_x u_x)\, dx$$

$$+ \int (p_2v_2u_2')\, dx$$

$$+ \sum_{i=1}^{2} \int (\tau_i^* v_i) u_i\, dx$$

$$= \{v_1(b)\}^* C_1[u_x](b) - \{v_2(b)\}^* C_2[u_2](b)$$

$$= \{v_1(b)\}^* (CiA^{-1})A[u_1](b) - \{v_2(b)\}^* (CiB^{-1})B[u_2](b)$$

$$= \{K_1[v_1(b) - K_2[v_2(b)]* A[u_1](b).$$
Using the conditions \( f_a = 0 \) and \( f_c = 0 \) we have

\[
\hat{\sigma} = \left[ (P^2 V_2) u_1^2 + q_2 V_2 u_2 - (P^2 V_2)' u_2 \right](c)
\]

\[
-\left[ (P^2 V_2) u_1^2 + q_1 V_1 u_1 - (P^2 V_2)' u_1 \right](a)
\]

\[
= u_1(a) P^*_a - u_2(c) P^*_c,
\]

where

\[
f_c = p x(a) \tilde{x}(a) - v x(a) [v_x(a) - p^a](a), \tag{7}
\]

\[
y_c = p_2(c) \tilde{v}_2(c) - v_2(c) [V c P_2(c) + (q_2(c) - p'_2(c))] \tag{8}
\]

With these simplifications, the Green's formula now becomes,

\[
(L\{u_1, u_2\}, \{v_1 V_2\}) = u_1(a) P^*_a - u_2(c) P^*_c
\]

\[
+ \left[ K_1[v_1(b)] - K_2[v_2(b)] \right] A [u_1(b)]
\]

\[
+ \int_{\Omega} \left( \tilde{z}_1 \tilde{y}_1 \right) u d x.
\]

**Lemma 1.** Let \( L \) be the operator given as in Eqs. (5) and (6) then its adjoint \( L^* \) is a densely defined closed unbounded operator given by

\[
D(L^*) = \{ [v_1, v_2] \in H / K_1[v_1(b)] - K_2[v_2(b)] \neq 0 \},
\]

\[
L^*[v_1, v_2] = \{ x^*_1 v_1 x^*_2 v_2 \}.
\]

**Proof.** Let \( T \) be the linear operator defined by

\[
D(T) = \{ [v_1, v_2] \in H / K_1[v_1(b)] - K_2[v_2(b)] \neq 0 \}.
\]

\[
T \{ v_1, v_2 \} = \{ x^*_1 v_1, x^*_2 v_2 \}.
\]

We claim that \( T = L^* \). For, clearly from Green's formula, it follows that \( D(T) \subseteq D(L^*) \).

To prove the inclusion \( D(L^*) \subseteq D(T) \), for \( \{ u_1, u_2 \} \in D(L) \) and \( \{ v_1, v_2 \} \in D(L^*) \), we must have

\[
\{ L[U_1, U_2], [V_1, V_2] \} = \{ [M_1, M_2], r \{ u_1, u_2 \} \},
\]

which follows once we show that

\[
U_1(a) P^*_a - u_2(c) P^*_c + (K_1[v_1(b)] - K_2[v_2(b)]) A[u_1(b)] = 0.
\]

\[
\text{For} \ \{ U_1, u_2 \} \in D(L) \ \text{such that} \ u_1(a) = u_2(c) = 0, \ \text{we get},
\]

\[
[K_1[v_1(b)] - K_2[v_2(b)]) A[u_1(b)] = 0.
\]

Green's formula then implies \( U_1(a) f^* \neq 112(0) P^*_c \).
Now for appropriate choice of functions \( \{u_1; u_2\} \in D(L) \) this can be shown to imply that \( J^*_p = J^*_g = 0 \).

This implies \( D(L^*) \subseteq D(T) \) and hence the lemma.

For our later use, we note here a few identities.

**Identity I.** For \( \{u_1, u_2\} \in D(L) \),

\[
(L[u_1, U_2], [u_1, U_2]) = \int (p_i W) + \int (P_2 W) Y_1 \begin{array}{c}
\rho \quad r \\
\|a \quad \|b
\end{array}
\int (z_2 U_2) U_2 dx
\]

\[= \int \left( (P_i W) + \int (P_2 W) Y_1 \right) + \int r_i |u_i|^2 dx
\]

\[= -p_i(a) v_i u_i(a)^2 + p_2(c) v_i \gamma_i u_i(c)^2
\]

\[+ p_1(b) u_1(b) u_i(b) - p_2(b) u_2(b) u_2(b)
\]

\[-\|J_2\|^2 \left\{ \int \rho \hat{M}^2 \, dx + \int \left( -p_1(a) v_i u_i(a) + p_2(c) v_i \gamma_i u_i(c) + \int r_i |u_i|^2 dx \right) \right\}
\]

**Identity II.** For \( \{v_1, v_2\} \in D(L^*) \),

\[
(L^* [v_1 U_2], [v_1 U_2]) = \int (p_i v_i X^-) + \int (q_i - p_i(v_i) v_i^2) \frac{d^p}{d_t}
\]

\[= \int \left( [p_2 v_i] + (q_2 - p_2 v_i) \right) v_i^2 dx
\]

\[= -\sum_{i=1}^2 \left\{ \int p_i |v_i|^2 dx + (q_i - p_i v_i)^2 + r_i f dx \right\}
\]

\[= -p_1(a) v_1(a)^2 + p_2(c) v_2(c)^2 + p_1(b) v_1(b) + p_2(b) v_2(b)
\]

\[-p_2(b) v_1(b)^2 - (q_1(b) - p_1(b)) v_1(b)^2
\]

\[+ r_i f v_i v_i + r_i v_i v_i dx
\]

\[-\sum_{i=1}^2 \left\{ \int p_i v_i v_i dx + (q_i - p_i f) v_i + r_i v_i^2 dx \right\}
\]
3. m-dissipativity of \((L - \lambda I)\)

We begin the section with the following definition which may be found in [5]. In what follows, \(X\) denotes a complex Hilbert space.

**Definition.** A linear, closed densely defined operator \(A:X \rightarrow \overline{D(A)}^X\) is called m-dissipative if:

(i) \(A\) is dissipative i.e. \(\text{Re}(\langle M,w \rangle) < 0, \forall w \in \overline{D(A)} \subseteq X\).

(ii) Range of \((XI - A)=X\), for some \(I>0\), (and hence for all \(I>0\)).

**Theorem 1.** The operator \((L - \lambda I)\), where \(L\) is as given in Eq. (6) is m-dissipative if

\[
(A^*J^*)D_2A^X = (B^*J^*)D_2B^X
\]

holds where \(co > 0\) is a real number, and \(D_i\) are matrices given by

\[
\begin{pmatrix}
\varphi(b) - p'iib \\
\rho(b) \\
i(b) \\
0
\end{pmatrix}
\]

**Proof.** Let

\[
\int_a^b (q - p')u' dx = Z \text{ (say)}
\]

Then a simple computation gives

\[
\text{Re}(Z) = 2I((a) - p(A))u(a)u(a) - (q(A) - p(A))u(a)u(a)
\]

\[
- \frac{1}{2} \left( \int_a^b (q_1 - p_1)u_1^2 dx \right)
\]

Similarly, we can show that

\[
\text{Re} \int_b^c (q_2 - p_2)u_2 u_2 dx = 12I((q_2(c) - p_2(c))u_2(c)u_2(c) - (q_2(b) - p_2(b))u_2(b)u_2(b)
\]

Now, one gets from Identity I that:

\[
\text{Re}(X[M_1,M_2],[M_1,M_2]) = \left[ \varphi_2(c)H - \frac{(q_1(c) - p_1(c))^2}{2} \right] u_2(c)u_2(c)
\]

\[
- \left[ \varphi_2(a) - \frac{(q_1(a) - p_1(a)}{2} \right] u_2(a)u_2(a)
\]
Firstly, we estimate the boundary terms at \( x = a \) and \( x = c \) as follows:

\[
\text{Boundary terms} = \left(p_2(c)v_c \left(\frac{q_2(c) - p_2(c)}{2}\right)\right)_{\text{u}} \mid_{x=c}^b + \left(p_2(c)v_c \left(\frac{q_2(c) - p_2(c)}{2}\right)\right)_{\text{u}} \mid_{x=a}^c
\]

where

\[
s_1(x) = (v_p p_1(a) + 2q_k(a) - p_k(a)) \left(\frac{b - x}{b - a}\right) \quad \forall x \in I_1.
\]

\[
s_2(x) = (v_p p_2(c) + 12q_2(c) - p_2(c)) \left(\frac{x - c}{c - b}\right) \quad \forall x \in I_2.
\]

We have,

\[
\left(s_1(x)\mid_{u_1(x)}\right)^2 = s_1'(x)\mid_{u_1(x)}^2 + 2s_1(x)\text{Re}(\overline{u_1(x)}u_1'(x))
\]

\[
\leq |s_1'|^2||u_1||^2 + 2|s_1||u_1||u_1'|
\]

\[
\leq |s_1'|^2||u_1||^2 + |s_1| \left(\text{Re}\lambda^2 + \text{Im}\lambda^2\right) \quad \text{for } \theta > 0.
\]

We thus have,

\[
\text{Boundary terms} \leq \sum_{i=1}^{2} \int_{I_i} \left(MM^2 + M\left(K\text{Re} + \frac{1}{\epsilon} V\right)\right) \quad \forall \epsilon > 0.
\]

Finally,

\[
\text{Re}\{L\{u_1, u_2\}, \{u_1, u_2\}\} \leq \frac{1}{2} \text{Re}\{\|A[u_1]\|_b D, \|A^{-1}\|_b D, \|B[u_2]\|_b D\}
\]

\[
- \frac{1}{2} \sum_{i=1}^{2} \int_{I_i} \left|u_i\right|^2 \left\{q_i(p_i - p_i^2 - 2r_i) - |u_i^2| + |s_i|e^{-2}\right\} \quad \text{dx}
\]

\[
- \sum_{i=1}^{2} \int_{I_i} \left|u_i\right|^2 \left|p_i - |s_i|e^2\right| \quad \text{dx}.
\]
For sufficiently small $\delta > 0$, such that $\{ p_i - \|y_i\|e^2 \} > 0$, we get,

$$\Re \{ L\{u_1, u_2\}, \{u_1, u_2\} \} \leq \sum_{i=1}^{2} \int_{\Omega} |u_i|^2 \left( t - q_i^i + p_i^i + 2r_i + \left( |y_i^i| + |y_i e^{-2}| \right) \right) dx^*$$

$$\leq w(\|y_1\|^2 + \|y_2\|^2) = \alpha \|\{ u_1, u_2\}\|^2$$

where, $\alpha = \max \sup \{ \sqrt{(-q_i^i + p_i^i + 2r_i + (|y_i^i| + |y_i e^{-2}|)} \}$. Thus, we have shown,

$$\Re \{ L\{u_1, u_2\}, \{u_1, u_2\} \} \leq O\|\{ u_1, u_2\}\|^2,$$

which implies $(L - \text{col})$ is dissipative.

Now, for showing $(L - \text{col})$ is $m$-dissipative we show that $(L^* - \text{OII})$ is dissipative. We get in an exactly similar manner,

$$\Re \int_{\Omega} (q_1 - p_1^i) v_1^i e^iv_1^i dx = \frac{1}{2} \left( q_1(b) - p_1(a) \right) - (q_1(a) - p_1(a)) v_1^i$$

$$- \frac{1}{2} \int_{\Omega} (q_1^i - p_1^i) |v_1| dv,$$

$$\Re \int_{\Gamma} (q_2 - p_2^i)v_2 dx = 12(q_2(c) - q_2(c))|v_2(c)|^2 - (q_2(b) - p_2(b)) v_2(b)$$

$$- \int_{\Gamma} (q_2^i - p_2^i)v_2 dx,$$

$$\Re \{ L^*\{v_1, v_2\}, \{v_1, v_2\} \} = \left( -p_1(a)v_a - \frac{(q_1(a) - p_1(a))}{2} \right) |v_1(a)|^2$$

$$+ \left( p_2(C)v_c + \frac{(q_2(c) - p_2(c))}{2} \right) |v_2(c)|^2$$

$$- \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega} (q_i^i - p_i^i - 2r_i) |v_i|^2 dx - \sum_{i=1}^{2} \int_{\Omega} p_i |v_i|^2 dv$$

$$+ (q_2(b) - p_2(b)) v_2(b) - (q_2(b) - p_2(b)) v_2(b)$$

$$+ p_2(b)v(b)v(b) - p_2(b) v_2(b)$$

$$= \text{Boundary terms + Interface terms + Integral terms.}$$
Now, with same choice of \( s_i(x) \) and \( s_2(x) \), we get

\[
\text{Boundary Terms} = \int_a^b (s_1(x) | v(x)\nabla x^2)dx + \int_a^b (s_2(x) | v_2(x)\nabla x^2)' dx
\]

\[
\leq \sum_{i=1}^2 \int_I |s_i'|^2 + \nabla K\nabla x^2 + \nabla K e^{-2})dx.
\]

The interface terms can be made to vanish using the condition \((K^{-1}_i \ast \ast (K T_i) = (K^2 T_i) \ast D_2 K T_i,\) which itself is a consequence of \((A^\ast) \ast D \ast (A^\ast) = (B^2 \ast D \ast B^\ast) \ast (6)\).

Finally, choosing a sufficiently small \( \delta > 0, \)

\[
\text{Re}\{L^\ast \{v_1, v_2\}, \{v_1, v_2\}\}
\]

\[
\leq \frac{1}{2} \text{Re}\{-v_1(b) \ast D_1[v_1](b) + [v_2(b)] \ast D_2[v_2](b)\}
\]

\[
- \frac{1}{2} \sum_{i=1}^2 \int_I [q_i - p_i^\ast + 2r_1]v_1^2 dx - \sum_{i=1}^2 \int_I p_i |v_i|^2 dx
\]

\[
+ \sum_{i=1}^2 \int_I [k_i' |v_i|^2 + |s_i| (|v_i|^2 e^{-2} + |v_i|^2 e^{-3})] dx
\]

\[
= \frac{1}{2} \sum_{i=1}^2 \int_I (-q_i' + p_i^\ast + 2r_1) + |s_i| + e^{-2} |s_i| |v_i|^2 dx
\]

\[
- \frac{2}{2} \sum_{j=1}^2 \int_I (|s_j|^2 (p_j - |s_j|^2)) dx
\]

\[
\leq \sum_{i=1}^2 \int_I (-q_i' + p_i^\ast + 2r_1) + |s_i|^2 + e^{-2} |s_i|^2 |v_i|^2 dx.
\]

So, we have shown,

\[
\text{Re}\{L^\ast \{v_1, v_2\}, \{v_1, v_2\}\} < \text{O}\{V_0\},
\]

where \( ! \) is same as before.

4. Analyticity of the semigroup generated by \( L \)

Here, we show under certain assumptions on the matrices \( A \) and \( B \) that the operator \( (L - \text{col}) \) generates an analytic semigroup of contractions. We establish the main result of this section with the help of the following theorem.
Theorem 2 (Fattorini [3]). Let $A$ be a densely defined operator in a complex Hilbert space $X$ such that, for $u \in D(A)$,
\[ \Re\{Au,u\} + \Re\{Au,u\} \leq 0. \]

Then $A$ generates an analytic semigroup of contractions.

Our main result is the following.

Theorem 3. The operator $L$ generates an analytic semigroup if the following conditions hold:
\[ (A^{-1})^*CiA^{-1} = (B^{-1})^*C_2B - \lambda \]
\[ (A^{-1})^*A^{n+1} = (-8)^{n+1}D_2 \cdot 8^{n+1}, \]
where the matrices $Q$ and $D_i$ are as given before.

Proof. From the definition of $m$-dissipativity, it is clear that $R(XI - A) = X$, where $A = (L - mL)$. It only remains to be shown that $(L - mL)$ generates an analytic semigroup. For that, we show that $A = (L - mL)$ satisfies
\[ \Re\{Au,u\} + S \Re\{Au,u\} \leq 0 \]
by showing
\[ \Re\{Lu,u\} + S \Im\{Lu,u\} \leq \|U\|^2 \]
hold for all $u \in D(L)$ under the given assumptions. From the Identity I,
\[ (5)\Re\{L[ui,u_2],[ui_1,u_2]\} \leq 8\Re\{p_i(b)[u_i(b)^{(2)}] - p_2(b)[u_2(b)^{(2)}]\} \]
\[ + 8 \left| \sum_{i=1}^{N} \int_{A_i} r_i \cdot q_i \cdot \left| u_i\right| ^2 \right| . \]

But, under assumption (10) and the interface conditions, we have,
\[ \Re\{Lu_1(b), u_2(b)\} = \Re\{P, Pi(b) - p_2(b)[u_2(b)]\} = 0 ; \]
\[ \Re\{Lu_2(b), u_1(b)\} = \Re\{P, Pi(b) - p_2(b)[u_2(b)]\} = 0 ; \]
\[ = \Re\{L[ui,b], [ui_1,b]\} \Rightarrow \Re\{Lu_1(b), u_2(b)\} = \Re\{L[ui,b], [ui_1,b]\} \Rightarrow \Re\{Lu_1(b), u_2(b)\} = 0 ; \]
\[ \Re\{Lu_2(b), u_1(b)\} = \Re\{L[ui,b], [ui_1,b]\} \Rightarrow \Re\{Lu_2(b), u_1(b)\} = 0 ; \]
\[ = 0 ; \]
We also have,

\[ \text{Im} \int_{I_i} (q_i - p_i')u_i u_i' dx = \left| \int_{I_i} (q_i - p_i') \text{Im} (\bar{u}_i u'_i) dx \right| \]

\[ \leq \int_{I_i} (q_i - p_i') |(\varepsilon^{-2}|u_i|)^2 + \varepsilon^2 |u_i'|^2 | dx \]

So that, finally we could write, for a sufficiently small \( \varepsilon \),

\[ \text{Re} \{ L(u, u_2), (u, u_2) \} + 5 \text{Im} \{ L(u, u_2), (u, u_2) \} \]

\[ \leq -\sum_{i=1}^{2} \int_{I_i} |u_i|^2 \left\{ (q_i' - p_i'' - 2r_i) - (|s_i'| + |s_i| |s_i^{-2}|) \right\} dx \]

\[ -\sum_{i=1}^{2} \int_{I_i} |q_i - p_i'| (|s_i| \varepsilon^{-2}) dx \]

\[ + \sum_{i=1}^{2} \int_{I_i} \left\{ (|q_i| + |s_i| \varepsilon^{-2}) - p_i \right\} |u_i|^3 dx \]

\[ \leq \sum_{i=1}^{2} \int_{I_i} |u_i|^2 \left\{ (-f_i' + p_i'' + 2r_i) + (|s_i'| + |s_i| \varepsilon^{-2}) \right\} dx \]

\[ -\sum_{i=1}^{2} \int_{I_i} |q_i - p_i'| |s_i| \varepsilon^{-2} dx \]

\[ \leq \omega \left\{ \|u_1, u_2\| \right\}^2 \]

for the same \( \omega \) as in the previous section. Thus, we have shown that \((L_{col})\) satisfies the required inequality. Hence the theorem.

5. Physical example

Let \( I_1 = [a, b] \) and \( I_2 = [b, c] \) : \( \infty < a < b < c < + \infty \) We consider the following parabolic initial boundary value problem:

\[ \frac{du}{dt} = a(x) \frac{\partial^2 u}{\partial x^2} \left. \right|_{I_i} \text{ on } I_i, \quad t = 1, 2, \]

where \( a_i(x) \) are continuously differentiable functions on \( I_i \); \( i = 1, 2 \).
At the interface \( x = b \) for \( t > 0 \), we have

\[
k_1 \frac{\partial u_1(x,t)}{\partial x} = k_2 \frac{\partial u_2(x,t)}{\partial x}
\]

(11)

\[
- k_1 \frac{\partial u_1(b,t)}{\partial x} = k(u_1(b,t) - u_2(b,t))
\]

(12)

The end point conditions are taken to be

\[
\frac{\partial u}{\partial x}(c,t) = 0
\]

(13)

\[
\frac{du}{dc}(c,t) = v_u u_2(c,t)
\]

(14)

And the initial conditions are given by

\[
u_i(x) = f_i(x) \quad \text{on } 4 \quad j = 1, 2.
\]

(15)

where \( k, k_1, k_2 \) are positive constants and \( v_u, v_c \) are some real constants.

Such problems arise in the study of heat conduction with linear thermoelasticity in composite rods. For example, they can be found as a special case of the type of problems discussed in [1].

The matching interface conditions can be written in the form, \( A[u_1](b) = B[u_2](b) \), where the matrices \( A \) and \( B \) are of the form.

\[
\begin{pmatrix}
0 & \lambda \\
\lambda & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & k_2 \\
k_1 & 0
\end{pmatrix}
\]

We consider the following abstract Cauchy problem:

\[
\frac{dU}{dt} = L U(t), \quad t > 0, \quad U(0) = U_0
\]

where \( U(t) \) is a \( X \times L^2(I_1) \times L^2(I_2) \) valued function, and the operator \( L \) is given by,

\[
D(L) = \{uu_1u_2eH/A[u_1](b) = B[u_2](b), uu'_1(a) = u_1(a)v_2, uu''_2(c) = v_2u_2(c)\},
\]

\[
L[u_1, u_2] = \{a(x)u'_1, cx^2(x)u''_2\}.
\]

It can be easily verified that the assumptions of the Theorem 3 are satisfied if

(i) \( a(x) > 0 \) and (ii) \( a(b) = a'(b) \). (iii) \( k_1 = k_2 = a(b)a_2(b) \). Thus, whenever the conditions (i)-(iii) hold, and \( f_2 \in D(L) \) then the abstract Cauchy problem and (hence) the parabolic initial boundary-value problem (11)—(15) has a unique solution which is analytic in time for \( t > 0 \).

References