Thermoeconomic optimization of an irreversible Stirling cryogenic refrigerator cycle

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Abstract

This communication presents the thermoeconomic optimization of an irreversible Stirling cryogenic refrigerator cycle. The external irreversibility is due to finite temperature difference between the working fluid and the external reservoirs while the internal irreversibility is due to the regenerative heat loss. The thermoeconomic function is defined as the cooling load divided by the total cost of the system plus the running cost. The thermoeconomic function is optimized with respect to the working fluid temperatures and the values for various parameters at the optimal operating condition are calculated. The effects of different operating parameters on the performance of the cycle have been studied. It is found that the effect of regenerative effectiveness is more than those of the other parameters on all the performance parameters of the cycle, for the same set of operating condition.

Keywords: Cryogenics; Thermodynamic cycle; Stirling; Experiment; Optimization; Capacity; Energy balance; Energy consumption

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Mots-clés: Cryogénie; Cycle thermodynamique; Stirling; Expérimentation; Optimisation; Puissance; Bilan énergétique; Consommation d'énergie

1. Introduction

The Stirling cycle is one of the important models of refrigeration systems for the production of very low temperature which has promoted the new developments in the design of this cycle. It is also desirable to have the corresponding results [1] for the coefficient of performance (COP) of the refrigeration systems. Leff and Teeters [2] have noted that the straight-forward Curzon-Ahlborn [1] calculations will not work for a reversed Carnot cycle because there is no 'Natural Maximum' in these systems. Blanchard [3] has applied the Lagrangian method of undetermined multiplier to find out the COP of an endoreversible Carnot heat pump operated at minimum power input for a given heating load.

In recent years the performance of the Carnot, Brayton, Stirling, Ericsson and other refrigeration cycles has been investigated by a number of researchers using the concept of finite time thermodynamics [4-9], the ecological criteria [10-17] and thermoeconomic approach [18-21] for different operating conditions.

In this paper we will investigate the thermoeconomic
optimization along with a detail parametric study of an irreversible Stirling cryogenic refrigerator cycle for different operating conditions using the concept of finite time thermodynamics [1].

2. System description

It is well known that the working substance of the Stirling cycle may be a gas, a magnetic material etc. For different working fluids, this cycle has different performance characteristics. When the working substance of the cycle is the perfect/ideal gas the cycle consists of two isothermal and two isochoric processes as shown on schematic and T-S diagrams in Fig. 1. This cycle approximates the expansion stroke of a real cycle as an isothermal process 1-2 with an irreversible isothermal heat addition at temperature \( T_c \) from a heat source of finite heat capacity whose temperature varies from \( T_{L1} \) to \( T_{L2} \): The heat addition to the working fluid from the regenerator is modeled as an isochoric process 2-3. The compression stroke is modeled as an isothermal process 3-4 with an irreversible heat rejection at temperature \( T_h \) to the heat sink of finite heat capacity whose temperature varies from \( T_{H1} \) to \( T_{H2} \). Finally, the heat rejection from the working fluid to the regenerator is modeled as an isochoric process 4-1, thereby completing the cycle.

As mentioned earlier, the heat transfer processes 1-2 and 3-4 in a real cycle must occur in finite time. This requires that these heat processes must be proceeded through a finite temperature difference and therefore, defined as being externally irreversible. There is also some heat loss through the regenerator, as an ideal regeneration requires an infinite regeneration time or area, which is not the case in practice.

3. Thermodynamic analysis

Let \( Q_c \) be the amount of heat absorbed from the source at temperature \( T_c \) and \( Q_h \) be the amount of heat released to the sink at temperature \( T_h \) during two isothermal processes, then:

\[
Q_c = T_c S_c = C_L(T_{L1} - T_{L2}) \tau_L = U_L A_L (\Delta T_{L1} - \Delta T_{L2})
\]

\[
Q_h = T_h S_h = C_L(T_{H1} - T_{H2}) \tau_H = U_H A_H (\Delta T_{H1} - \Delta T_{H2})
\]
where $\Delta S = nR_0 \ln lP$ is the entropy change, $l$ is the volume ratio, $n$ is the number of mole for the working fluid, $R_0$ is the universal gas constant, $C_L, CH$ and $t_L, t_H$ are the heat capacitance rates of source/sink reservoirs and heat addition/rejection times, $t_L$ and $t_H$ are the effectiveness of the heat exchangers for hot- and cold-side, respectively, $U_0A_H$ and $U_0A_C$ are the overall heat transfer coefficient-area products and $(LMTD)_H$ and $(LMTD)_C$ are the log mean temperature difference on sink- and source-side, respectively, defined as:

$$\Delta LM T D _ { H } = [ ( T _ { H } - T _ { H } l ) - ( T _ { H } - T _ { H } l ) ] / l n [ ( T _ { H } - T _ { H } l ) ]$$

$$\Delta LM T D _ { C } = [ ( T _ { C } - T _ { C } l ) - ( T _ { C } - T _ { C } l ) ] / l n [ ( T _ { C } - T _ { C } l ) ]$$

Using Eqs. (1) and (2) and Eqs. (3) and (4), we can get:

$$Q_H = C_L t_l U T _ H T _ L / t _ H$$

$$Q_C = C_H t_H U T _ L T _ H / t _ L$$

where $t_L = 1/2 \frac{n R_H C_H}{U A} \Delta T_H$ and $t_H = 1/2 \frac{n R_C C_L}{U A} \Delta T_C$ are the effectiveness of the hot- and cold-side heat exchangers, respectively.

As these cycles, in general, do not possess the condition of perfect regeneration, it is assumed reasonably that the regenerative heat loss per cycle is proportional to the temperature difference of the two isothermal processes as assumed by earlier workers [5-9].

$$A g _ { R } = n c_f ( T _ H - T _ C ) ( T _ H - T _ C )$$

where $c_f$ is the molar specific heat capacity of the working fluid and $1_R$ is the effectiveness of the regenerator, defined as:

$$k _ R = \frac{Q _ { gen _ { actual } }}{Q _ { gen _ { ideal } }} = \frac{Q Y _ H T _ H T _ L}{T Y _ H \cdot 2 T _ L} = \frac{N _ R N _ R}{1 + N _ R}$$

where $N_R = \frac{U A H P_c}{C_f}$ is the number of transfer unit of the regenerator and $C_f$ is the heat capacitance rate of the working fluid.

Owing to the influence of irreversibility of finite heat transfer, the regenerative time should be finite as compared to that of the two isothermal processes [5-9,17]:

$$t_e = \frac{1}{2} \frac{a}{k} T _ H - T _ C$$

where $a$ is the proportionality constant, which is independent of the temperatures on the hot- and cold-side but depends on the property of the regenerative material. Thus, the total cycle time $t_{cycle}$ will be:

$$t_{cycle} = t_H \cdot t_H \cdot t_H \cdot t_H$$

When the irreversibilities mentioned above are taken into account, the net amounts of heat released to the sink and absorbed from the source are:

$$q_u = q_h - \Delta Q _ H$$

$$q_L = q_C - \Delta Q _ C$$

For a Stirling refrigeration cycle, the irreversibilities associated with the heat transfer during the compression and expansion of the working fluid are a significant source of thermodynamics inefficiencies of the system. Thus we have:

$$Q = \frac{2}{1} \frac{1}{S _{gen}}$$

where $T_u$ and $T_L$ are, respectively, the average temperatures of the sink- and source-fluid at constant pressure and $S _{gen}$ is the entropy generation rate of the cycle. Using Eqs. (1), (2), (11) and (12), we have:

$$Q / T _ H - Q / T _ L = 0$$

$$W = Q_u - Q_L = q_h - q_C$$

It is worthwhile to note that for an irreversible Stirling cycle Eq. (14) is true, although the irreversibilities associated with the heat transfer are considered.

Since, the power input, cooling load and COP are the important parameters of a refrigerator cycle. Using Eqs. (11)-(15), we have:

$$R _{u} = \frac{Q _{u}}{\eta_{cycle}} = \frac{\eta _{r}}{\eta _{h} + \eta _{h} + \eta _{r}}$$

$$P = \frac{W}{\eta_{cycle}} = \frac{(1 - \eta _{r}(x - 1))}{\eta _{h} + \eta _{h} + \eta _{r}}$$

$$\frac{R _{u}}{P} = \frac{Q _{u} - \Delta Q _ H}{Q _{h} - Q _{C}} = \frac{1 - \eta _{r}(x - 1)}{(x - 1)}$$

where $k_H = 1 H C H, k_L = 1 L C L, x = T _ H / T _ L, y = T _ C, b_l = 2 a = \partial n \cdot \ln lP$ and $a_l = \eta _{r} 2 l s p = R _{0} \ln l$.

The objective function of thermoeconomic optimization proposed by earlier workers [18-21] is given by:

$$F = \frac{1}{C_i + C_e}$$

where $C_i$ and $C_e$ refer to annual investment and energy consumption costs, respectively. The investment cost was considered the costs of the main system components that are the heat exchangers and the compression and expansion devices together. The investment cost of the heat exchangers is assumed to be proportional to the total heat transfer area [18-21]. On the other hand, the investment cost due to the compression and expansion devices is assumed to be proportional to their compression/expansion capacities or...
the required power input. Thus the annual investment cost of the system can be given by:

\[ C_i = o(A_{HI} + A_{UL} + A_H) + b_3 P \]

where the proportionality constant for the investment cost of the heat exchanger, \( a \), is equal to the capital recovery factor times investment cost per unit heat exchanger area and the proportionality constant for the investment cost for the compression and expansion devices, \( b_3 \), is equal to the capital recovery factor times investment cost per unit power input. The initial investment cost is converted to equivalent yearly payment using capital recovery factor [18-21]. 

Thus optimizing the economic parameter \( \frac{E}{Pe} \) with respect to \( y \) yields:

\[ \frac{dF}{dy} = 0 \]

Substituting Eqs. (20) and (21) into Eq. (19), we have:

\[ F = \frac{\dot{Q}_L}{a(\dot{A}_{HI} + \dot{A}_{UL} + \dot{A}_H) + b_3 \dot{Q}_H, \dot{Q}_c = \dot{Q}_h} \]

where the coefficient, \( b_3 \), is equal to the equivalent annual operation hours corresponding to power input times price per unit energy [18-21]. Thus the annual energy consumption cost is proportional to the power input.

\[ C_e = b_4 P = b_3 \dot{Q}_H, \dot{Q}_c = \dot{Q}_h \]

Substituting Eq. (24) into Eqs. (23), (16) and (17), we have:

\[ bF = \frac{x}{k_4 k_1 k_3 (1 + e_{12})} \]

where \( k_4 = \frac{\eta_{H} R_{12}}{\eta_{H} L_{12}}, k_4 = (1 + e_{12})(e_{12} + k_2 k_3), k_4 = k_1 k_2 k_3 \) and \( b_4 = 1 \cdot k_2 k_3. \)

4. Discussion of results

In order to have the numerical appreciation of the results for the Stirling refrigerator cycle, we continue to investigate the effects of the effectiveness of different heat exchangers \((1_{1H}, 1_{1L}, 1_{2H})\), the economic parameter \( \frac{\partial k_1 P}{k_1} \) and heat capacitance rates \((C_{H1} \text{ and } C_{H2})\). The effects of each one of these parameters are examined while the rest of the parameters are kept constant as \((k_{1H} = 1_{1L} = 1_{1R} = 0.80, \quad 1_{1H} = 290 \text{ K}, \quad 1_{1L} = 200 \text{ K}, \quad 1_{2H} = 0.50, \quad 1 = 2.0, \quad C_{H1} = C_{H2} = 10 \text{ kW/K}, \quad b_4 = 0.01, \quad U_{HI} = U_{HL} = U_{HR} = 2.0 \text{ kW/m}^3)\), the results obtained are as follows.

4.1 Effects of cycle temperature ratio \((x)\)

The variation of the objective function, COP, power input and cooling load with respect to the cycle temperature ratio \( \frac{\partial x}{T} = \frac{T_c}{T} \) for a typical set of operating parameters is shown in Fig. 2. It is seen from Fig. 2 that the objective function and the cooling load first increase and then decrease while the COP monotonically decreases whereas the power input monotonically increases as the cycle temperature ratio increases. These properties may be directly expounded by Eqs. (25)-(27) and (18), because the objective function and cooling load are not monotonic functions of \( x \) while the power input and COP are respectively, the monotonically increasing and decreasing functions of \( x \). It can also be clearly seen from Fig. 2 that both the objective function and the cooling load attain their maxima but at different value of \( x \) and there exists the
following relation:

\[ \delta_y^{opt \beta F} \overset{\neq 0}{\neq} \delta_y^{opt \beta R_L} \]

where \( \delta x^{opt \beta F} \) and \( \delta x^{opt \beta R_L} \) represent the two different optimal values of \( x \); the former one corresponds to the point of the maximum objective function while the later one corresponds to the point of the maximum cooling load, respectively. It is seen from Eqs. (25) and (26) that both the performance parameters, i.e. \( bF \) and \( R_L \) are the functions of a single variable, \( x \); for a typical set of operating condition. Thus, maximizing \( bF \) and \( R_L \) with respect to \( x \) yields:

\[ (x_{opt}^{\beta})_{RG} = \frac{B \pm \sqrt{B^2 - AC}}{A} \]

where \( A = \frac{1}{2} T_{11} + a_2 T_{11} \), \( B = b_2 T_{11} T_{11} + C = \frac{1}{2} T_{11} + a_2 T_{11} \), \( A_1 = \frac{1}{2} T_{11} + a_2 T_{11} \), \( B_1 = b_1 T_{11} T_{11} \) and \( C = \frac{1}{2} T_{11} + a_2 T_{11} \).

It can be seen from Eqs. (29) and (30) that both the parameters, i.e. \( \delta x^{opt \beta F} \) and \( \delta x^{opt \beta R_L} \), have two roots for a given set of operating parameters, but the roots containing the minus sign do not give the useful results. So we calculated the values of \( \delta x^{opt \beta F} \) and \( \delta x^{opt \beta R_L} \) only with the plus sign. Substituting the values of \( \delta x^{opt \beta F} \) into Eqs. (25)-(27) and (18) we can calculate the maximum objective function and the corresponding cooling load, power input and COP, while the maximum cooling load and the corresponding power input and COP can be calculated by substituting \( \delta x^{opt \beta R} \) into Eqs. (18) and (27). On the other hand, the optimal values of \( y \) for both the cases, i.e. \( \delta y^{opt \beta F} \) and \( \delta y^{opt \beta R_L} \) can be calculated by substituting the values \( \delta y^{opt \beta F} \) and \( \delta y^{opt \beta R_L} \) separately, into Eq. (24) for a typical set of operating condition.

4.2. Effect of effectiveness

Fig. 3(a)-(d) show the effects of the effectiveness on the maximum objective function, the corresponding COP, cooling load and power input of an irreversible Stirling refrigeration cycle. (a) Maximum objective function vs. effectiveness. (b) Corresponding COP vs. effectiveness. (c) Corresponding cooling load vs. effectiveness. (d) Corresponding power input vs. effectiveness.
cooling load and power input of an irreversible Stirling cryogenic refrigeration cycle. It is seen from these figures that the maximum objective function and the corresponding COP first increase and then decrease while the corresponding cooling load and power input increase as the effectiveness on any heat exchanger increases. Their physical meaning may be explained as follows. For the typical set of operating parameters given above, it implies the fact that the larger the effectiveness is, the larger the heat transfer areas are required and the smaller the irreversibility associated with the cycle is. This results in the larger values of the corresponding cooling load and power input. In general, the corresponding cooling load is not a linear function of the corresponding power input, so that there may be a maximum for the COP. On the other hand, when the heat transfer areas are increased, the corresponding cooling load and the cost of the system increase. In general, the corresponding cooling load is not a linear function of the cost of the system, so that there may be a maximum for the objective function. As \( I \rightarrow \infty \), so the size of the system and hence, the cost becomes very large. This results a sharp increase in the corresponding cooling load and power input while a sharp decrease in the objective function and the corresponding COP as can be seen from Figures 3(a-d). Again, as the effect of the internal irreversibility is more than that of the external one and the regenerative-side effectiveness belongs to the internal one unlike the hot- and cold-side effectiveness, which belongs to the external one. Hence, the effects of the regenerative effectiveness are more than those of the hot- and cold-side effectiveness on all the performance parameters for the same set of operating condition. Moreover, the effects of the hot- and cold-side effectiveness are almost the same for all performance parameters, so these two curves overlap, as can be seen from these figures.

4.3. Effect of \( k_1 \)

The effects of economic parameter \( k_1 \) on the maximum objective function, the corresponding cooling load, power input and COP are shown in Fig. 4(a) and (b). It is seen from these figures that the maximum objective function and the corresponding COP decrease while the cooling load and the power input increase as the economic parameter \( k_1 \) increases. Since, the cost of the system increases by increasing the economic parameter so the maximum objective function decreases with increasing the economic parameters. Also the optimal value of cycle temperature ratio \( \frac{1}{x_{opt}} \) increases with increasing \( k_1 \), resulting in an increase in the corresponding cooling load and power input but a decrease in the objective function. Since, the corresponding power input in this region increases more than that of the corresponding cooling load, hence, as a result the COP decreases, by increasing the economic parameter, which can be seen clearly from these figures.

4.4. Effect of heat capacitance rates

The effects of source/sink-side heat capacitance rate on the maximum objective function, the corresponding COP, cooling load and power input are shown in Fig. 5(a) and (b). It is seen from these figures that the maximum objective function as well as the corresponding COP decrease, as the heat capacitance rate on source/sink-side reservoir increases. On the other hand, the corresponding cooling load or power input is found to be a monotonically increasing function of the heat capacitance rate on the source/sink-side. These results can be explained as follows.

For the typical set of operating parameters given above, it implies the fact that the larger the heat capacitance rates are, the larger the heat transfer areas are required. This results in the larger values of the power input, cooling load and the cost of the system. As a result the maximum
Fig. 5. (a) and (b) Effects of the source/sink-side heat capacitance rate \( (C_H \text{ or } C_L) \) on the maximum objective function and the corresponding COP, cooling load and power input of an irreversible Stirling refrigeration cycle. (a) Maximum objective function and optimal COP vs. heat capacitance rate. (b) Corresponding cooling load and power input vs. heat capacitance rate.

objective function and the corresponding COP decrease while the cooling load and power input increase, as the heat capacitance rates of the external fluid increases.

5. Conclusions

A more realistic Stirling refrigerator cycle model including external and internal irreversibilities for the finite heat capacities of external reservoirs has been studied in detail. The external irreversibilities are due to finite temperature difference between the cycle and the external reservoirs, while the internal irreversibility is due to the heat loss inside the regenerator. The thermoeconomic function, which is defined as the cooling load per unit cost of the system (viz. running, investment etc.) has been adopted as an objective function for maximization. The objective function is maximized with respect to the cycle temperatures and the corresponding cooling load, power input and COP are evaluated for the different operating conditions. The objective function is found to be a decreasing function of the economic parameter and heat capacitance rate while there are optimal values of the cycle temperature ratio \( \delta x_p \) and different effectiveness, at which the objective function attains its maximum value for a typical set of operating condition. It is also found that effects of the regenerative-side effectiveness is more than those of the other side effectiveness, which can be explained in terms of internal and external irreversibilities. On the other hand, the effects of the source- and sink-side parameters are found to be similar, so the two curves overlap, as can be seen from graphs. Thus, the present cycle model gives some optimal criteria, which will be useful to understand and optimize the design performance of a real cycle from the point of view of thermodynamics as well as from the point of view of economics. The results obtained here are also applicable to the Ericsson cycle, in which the only difference is that the two isochoric processes will be replaced by the two isobaric processes and the volume ratio will be replaced by pressure ratio and hence, the specific heat at constant volume will be replaced by the specific heat at constant pressure.

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References


