Calculation of Cutoff Frequencies in Optical Fibers for Arbitrary Profiles Using the Matrix Method

ENAKSHI K. SHARMA, I. C. GOYAL, AND AJOY K. GHATAK

Abstract—We here propose a simple numerical procedure to calculate the cutoff frequencies in optical fibers with any arbitrary refractive index profile including discrete numerical data from profile measurement. The cutoff problem is transformed into a matrix eigenvalue problem and the cutoff frequencies can be obtained by determining the eigenvalues of a matrix with elements given by simple expressions.

INTRODUCTION

The cutoff frequency for single-mode operation is an important parameter since it defines an upper limit on the diameter of the fiber for single-mode operation. In general, except for a few refractive index profiles (like the step and the parabolic profile), analytical conditions defining the cutoff frequencies cannot be obtained and one has to use approximate techniques or direct numerical methods [11]. The approximate techniques are, however, restricted in their applicability; for example, the variation technique gives an accuracy of two percent only [12] while the perturbation methods give accurate results only for profiles which deviate slightly from the unperturbed profiles for which analytical solutions can be obtained [3]-[5]. Recently, approximate formulas for calculating the cutoff frequency of the TE_0 mode have been reported by Kokubun et al. [6]; these have been shown to give good accuracy, but involve successive approximations which require the evaluation of double and triple integrals at each stage. The numerical methods reported for directly solving the wave equation either involve extensive computations [7], [8] or are limited to specific functional forms of the profiles, e.g., the power series method [9] is limited to profiles which can be expressed as a finite power series.

In this paper we propose a simple numerical method for evaluation of the cutoff frequencies of modes in an optical fiber; the method is applicable to any arbitrary index profile including numerical data from index profile measurements. The cutoff problem is transformed into a matrix eigenvalue problem and the cutoff frequencies are evaluated as the eigenvalues of a matrix with elements given by simple expressions. The method has been primarily developed and tested for obtaining the single-mode limit, i.e., the cutoff frequency of the TE_0 mode. However, it is in general applicable to any mode, but can be used efficiently only for a few low-order modes; for higher order modes one has to work with large matrices to attain the required accuracies. Further, as an example the method has been used to determine the effect of an axial Gaussian refractive index dip on the single-mode limit of both step index and graded index optical fibers.

THEORY

The refractive index profile of a graded index optical fiber, in general, can be written in the form

\[ n^2(R) = n_1^2 + \delta (n_1^2 - n_2^2) f(R) \quad R \leq 1 \]

\[ = n_2 \quad R > 1 \]

where \( R = r/a, a \) being the radius of the fiber, \( n_1 \) is the maximum refractive index in the core, and \( n_2 \) is the cladding refractive index; \( f(1) = 1 \) and \( \delta \) defines the index discontinuity at the core cladding interface (\( \delta = 1 \) for a continuous profile). The scalar wave equation describing the field pattern in the core \( (R < 1) \) is given by

\[ \frac{d^2 \psi}{dR^2} + \frac{1}{R} \frac{d \psi}{dR} + \left( \alpha^2 - \alpha^2 f(R) \right) \psi = 0 \quad (2) \]

where \( \alpha \) is the propagation constant, \( k_0 \) is the free space wavenumber, and the \( \alpha \)-dependence of the field is of the form \( \exp (i k_0 R) \). We will restrict our analysis to the TE modes \( (l=1) \) since we are mainly interested in the TE_0 mode. For the TE modes, the field in the cladding is described in terms of the modified Bessel functions as

\[ \psi(R) = e^{-i k_0 R} K_l(w) \quad (3) \]

\[ w = k_0 a (n_1^2 - n_2^2)^{1/2} \]

\[ a = a(k_0 n_f - \delta^2)^{1/2} \]

\[ l = 0, 1, 2 \ldots \]

\[ \psi(0) = 0 \quad (5a) \]

\[ \frac{1}{K_l(w)} \frac{d \psi}{dR} \bigg|_{R=0} = w K_l(w) - 1 \quad (5b) \]

At the cutoff value of \( \alpha, \alpha^2 + k_0 \beta \); or \( \alpha + u \) and \( w + 0 \); (3) then reduces to (we have put \( l=1 \))

The method can be easily extended to other modes also.
and the boundary conditions reduce to:

\[ \psi(0) = 0 \]  
\[ \left( \frac{1}{\psi} \frac{d\psi}{dR} \right)_{R=1} = -1 \]  

The above differential equation and boundary conditions can be converted into the following integral equation [10] (see Appendix):

\[ R^2 \psi(R) + \int_0^R \xi \psi(\xi) d\xi = v_0^2 R^2 \left[ 1 - \delta f(R) \right] \left\{ \left( 1 - \frac{R}{2} \right) \int_0^R \xi \psi(\xi) d\xi \right\} + R \int_0^R \left( 1 - \frac{\xi}{2} \right) \psi(\xi) d\xi \]

Next, we replace the integrals in the above equation by a sum at discrete points chosen so as to divide the total domain (0 to 1) into N intervals. We have made use of the trapezoidal rule to evaluate the integrals, i.e., the integrand is approximated to a linear function in the interval 0 to 1/N. Doing so, we obtain N equations of the type

\[ R_i^2 y_i + \sum_{j=1}^{i-1} \frac{R_{ij} y_j}{N} + \frac{R_{ii} y_i}{2N} = v_0^2 R_i^2 \left[ 1 - \delta f(R_i) \right] \left\{ \sum_{j=1}^{i-1} \frac{R_{ij}}{2} \int_0^{R_i} \xi y(\xi) d\xi \right\} + \int_{R_{i-1}}^{R_i} \left( 1 - \frac{\xi}{2} \right) y(\xi) d\xi \]

where \( y_i = \frac{d^2 y}{dR^2} \). The above equations can be put in the matrix form as

\[ \begin{bmatrix} P & [y] \end{bmatrix} = v_0^2 \begin{bmatrix} 0 & 1 \end{bmatrix} \]

or

\[ [P^{-1}] \left[ y \right] = 1 \]

where \([P]\) and \([Q]\) are NX N matrices with elements defined by

\[ P_{ij} = R_i \frac{R_{ij}}{N} \]

\[ Q_{ij} = R_i \frac{R_{ij}}{2N} \]

For \( I > 1 \), the boundary condition at \( R = 1 \) becomes

Using the trapezoidal rule, one can write

\[ \int_0^{nh} f(x) dx = \left\{ \frac{f(0) + f(nh)}{2} + \sum_{k=1}^{n-1} f(kh) \right\} h. \]

and \([y]\) is a column matrix with elements \( y_i \). Hence, \( u^* \) can be evaluated by evaluation of the eigenvalues of the matrix \([P^{-1}Q]\), the largest eigenvalue corresponding to the \( \text{TE}_0 \) mode; the other values correspond to higher \( \text{TE} \) modes.

It may be mentioned that the following integral equation

\[ \psi(t) = v_0^2 \int_0^t G(t, s) s [1 - f(s)] \psi(s) ds \]

with

\[ Q^\alpha, \beta = \begin{cases} \frac{1}{2} & 0 < \alpha < 1 \\ \frac{1}{2} \alpha & 0 < \beta < 1 \end{cases} \]

obtained in [6] can also be transformed into a matrix equation by replacing the integral by a sum at discrete points. However, numerical calculations show that the convergence with respect to \( N \) is slower; this is expected since the integral (which we approximate by a linear function in any interval) contains the integrand instead of \( S^* \).

RESULTS AND DISCUSSION

To show the accuracy and convergence of the method, we carried out calculations for refractive index profiles for which the exact results are available. Various standard procedures are available for determining the matrix eigenvalues [11]. However, since we were mainly interested in the \( \text{TE}_0 \), mode and few low-order \( \text{TE} \) modes only, we used the power method for determining the eigenvalue largest in magnitude, and the "Wielandt's deflation" procedure to deflate the matrix for the second and third eigenvalue [11]. The results are shown in Table I. As can be seen, sufficient accuracy is attained for the first three modes in the parabolic index core profile for \( N = 30 \); the convergence is slower for the step profile. The cutoff value of the \( \text{TE}_0 \) mode, however, converges to the required accuracy for even smaller values of \( N \). The calculations were carried out on an ICL 2960 system; the time taken for the calculation of the first three eigenvalues even for the \( N = 30 \) case is less than 10 s.

Further, to illustrate the applicability of the method to any functional form of the profile we carried out calculations to show the effect of the central Gaussian refractive index dip on
TABLE 1

<table>
<thead>
<tr>
<th>a</th>
<th>Mode</th>
<th>Exact</th>
<th>( n_v ) from the matrix eigenvalue equation</th>
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<td>( l_m )</td>
<td></td>
<td>( J_1(V) ) ( W = 10 )</td>
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The exact results for the step profile \( a = -1 \) correspond to the zeros of the Bessel function of zeroth order \( J_0(M_1) \); for the parabolic profile \( a = 2 \) the results are taken from [141]. For all other values of \( a \) the results are taken from [5] which are based on the method in [191].

and \( p \) and \( d \) are parameters defining the dip depth and dip width, respectively. The dashed and continuous curves in Fig. 2 show the change in the single-mode limit of the radius \( a_d \) in the presence of the dip as compared to the limiting radius \( a_* \) in the absence of the dip for the step fiber and parabolic index fiber, respectively, for various values of \( p \) and \( d \). The effect is significant only for large dip widths; this is expected since the field of the TE, mode falls to zero around the axis. Further, the effect is much more significant in the graded index fiber than in the step index fiber.

**SUMMARY**

In summary, we have developed a simple matrix method for calculating the cutoff frequencies of modes in an optical fiber with an arbitrary refractive index profile. The method is also applicable to discrete refractive index data obtained from index profile measurements.

**APPENDIX**

To convert (6) into an integral equation we follow the procedure in [10], i.e., we define

\[
\gamma(R) = \frac{d^2 \psi}{dR^2}. \tag{A.1}
\]

Hence,

\[
\psi' = \psi' = \int_0^R \gamma(\xi) \, d\xi + C_1 \tag{A.2}
\]

or

\[
\psi = \int_0^R (R - \xi) \gamma(\xi) \, d\xi + C_1 R + C_2 \tag{A.3}
\]

where \( C_1 \) and \( C_2 \) are constants determined by the boundary conditions.

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The exact results for the step profile \( a = -1 \) correspond to the zeros of the Bessel function of zeroth order \( J_0(M_1) \); for the parabolic profile \( a = 2 \) the results are taken from [141]. For all other values of \( a \) the results are taken from [5] which are based on the method in [191].

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The continuous curve shows the refractive index profile of a parabolic graded index fiber in the presence of a Gaussian axial index dip given by (14). The dashed curve shows the profile in the absence of the dip.

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The single-mode limit of a step index and a parabolic index fiber. Earlier, such studies were reported on step index fiber only and most considered the central dip to be expressed by an inverse \( a \)-power law function [12]; recently, a Lorentzian dip on a step index fiber has also been considered [6]. It may be mentioned that the refractive index dip profile in practical fibers can be well represented by a Gaussian function [13]. In the presence of the dip the index profile of the fiber can be written as (the profile is shown in Fig. 1)

\[
n(R) = n(R) - \gamma(R) = n(R) - n(R) \left[ \frac{\gamma(R)}{\gamma(R)} \right] R \left[ \frac{n(R)}{n(R)} \right] R
\]

where \( n(R) \) is the profile in the absence of the dip and is given by

\[
\begin{align*}
n_0(R) &= \frac{1}{\sqrt{1 + \left( \frac{R}{R_0} \right)^2}} \\
&= \frac{1}{\sqrt{1 + \left( \frac{R}{R_0} \right)^2}} \quad \text{for the parabolic fiber} \\
&= \frac{1}{\sqrt{1 + \left( \frac{R}{R_0} \right)^2}} \quad \text{for the step fiber}
\end{align*}
\]

and \( \gamma(R) \) is the profile in the presence of the dip and is given by

\[
\gamma(R) = \frac{1}{\sqrt{1 + \left( \frac{R}{R_0} \right)^2}} \quad \text{for the parabolic fiber} \\
= \frac{1}{\sqrt{1 + \left( \frac{R}{R_0} \right)^2}} \quad \text{for the step fiber}
\]

The effect is significant only for large dip widths; this is expected since the field of the TE, mode falls to zero around the axis. Further, the effect is much more significant in the graded index fiber than in the step index fiber.
Fig. 2. The variation of the limiting radius for singlemode operation, \( a_o \), in the presence of the axial Gaussian refractive index dip; \( a_o \) is the limiting radius in the absence of the dip. The dashed and continuous curves correspond to the step and parabolic index fibers, respectively.

conditions (7a) and (7b). Equation (7a) implies \( C_2 = 0 \) and substituting \( S' \) and \( S \) from (A.2) and (A.3) into (7b) we get

\[
\frac{f^*}{f} \left[ 1 + \frac{\epsilon}{ \left( \frac{\epsilon}{2} \right)^2 } \right] \frac{d}{d\xi} \left( 1 - \frac{\xi}{2} \right) y(\xi) = 0.
\]

or

\[
\frac{y_1}{y_2} = \frac{\int_0^1 (1 - \xi) y(\xi) d\xi}{\int_0^1 (1 - \frac{\xi}{2}) y(\xi) d\xi}.
\]

Thus,

\[
\int_0^R \left( R - \xi \right) y(\xi) d\xi = \int_0^R \left( 1 - \frac{\xi}{2} \right) y(\xi) d\xi.
\]

Equation (6) can be written as

\[
R^2 y(R) + (R\psi' - \psi) = -v_c^2 \left[ 1 - \delta f(R) \right] R^2 \psi.
\]

From (A.2) and (A.3) (with \( C_2 = 0 \)) we can write

\[
R\psi' - \psi = \int_0^R y(\xi) d\xi.
\]

Substituting (A.9) and (A.7) into (A.8) we get the integral equation

\[
R^2 y(R) + \int_0^R y(\xi) d\xi =
\]

\[
v_c^2 R^2 \left[ 1 - \delta f(R) \right] \left\{ \left( 1 - \frac{R}{2} \right) \int_0^R \xi y(\xi) d\xi \right\} + R \int_0^R \left( 1 - \frac{\xi}{2} \right) y(\xi) d\xi.
\]

REFERENCES

Three-Waveguide Couplers for Improved Sampling and Filtering

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Abstract—Coupling between two uncoupled waveguides in proximity can be affected by introducing a third waveguide between them. One advantage of this scheme is that it can eliminate the need for bends. Another advantage is the resulting improvement in the shape of the transfer characteristic. Improved sampling and tunable filter operation of this new structure are also shown. An analysis of a uniform filter structure is presented. The ’’sidelobe’’ analysis of Alferness is adapted to the present case and tapers are investigated for further sidelobe suppression.

I. INTRODUCTION

In the design of planar guided wave optical systems incorporating conventional waveguide couplers [1]-[6] it is desirable to have relatively loosely confining waveguides in order to increase the width of single-mode guides, thereby simplifying their fabrication, and more importantly, in order to reduce the coupling length between coupled guides, thereby reducing system size. The waveguides in systems that use conventional two-guide couplers need to have bends. The bends impose the conflicting requirement for waveguides with relatively tight wave confinement so that bend radii can be small, and so that waveguides can be brought into proximity and separated over short distances, again in order to minimize system size. This conflict can be resolved by coupling the two waveguides by introducing a third waveguide between them. This three-guide coupler eliminates the need for bends in many systems, is compatible with loosely confining waveguides and short coupling lengths, and has a transfer characteristic that is in many respects superior to that of a two-guide coupler.

Coupled three-waveguide systems have been analyzed by Iwasaki et al. [7] and discussed further by Gauthier and Nurmikko [8]. The analysis is rather complicated and no general conclusions have been drawn by the mentioned authors. They also did not point out the improved transfer characteristics that can be achieved with properly designed three-waveguide couplers. In this paper, we analyze the three-waveguide coupler under certain symmetry conditions that greatly simplify the analysis and also lead to very desirable performance characteristics. We study a design for a sampler and show sampling characteristics that are improved versions of the two-waveguide coupler.

Alferness and Schmidt [9] have proposed and tested a design of a tunable two-waveguide optical filter. The design utilizes two unequal waveguides with dispersion curves (of the uncoupled waveguides) that cross at a frequency \( f_0 \). The frequency \( f_0 \) can be controlled by an applied voltage. At \( f_0 \), maximum power is transferred from one waveguide to the other; the power transfer versus frequency dependence determines the filter characteristic. Coupled waveguides with distance-in dependent coupling over a fixed length have filter characteristics with ”sidelobes.” The sidelobes can be partially suppressed by tapering the coupling (i.e., the distance between the two waveguides).

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