pling conditions, the process of measuring phase differences rather than phase does have experimental advantages if used carefully [41].

REFERENCES


Linear Phase IIR Filter with Equiripple Stopband
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Abstract—A simple technique of introducing transmission zeros into a low pass all-pole filter function to make its stopband transmission equiripple is described. The zeros are introduced through a transformation which maps the stopband of the filter onto the unit circle and the passband onto a segment of the real axis in the transformed plane. The technique is used to improve the loss characteristic of all-pole constant delay filters. Examples are included to illustrate the design procedure.

I. INTRODUCTION

Design methods of IIR filters with equiripple or maximally flat group delay have been outlined by Thiran [11, 2], Fettweis [31 and Deczky [4]. These filters are, however, all-pole functions with poor magnitude characteristic. Maria and Fahmy [5] have used optimization to improve the magnitude response of these filters. Their algorithm minimizes the mean-square error of the magnitude response of these filters from a desired response. We propose here a simple analytic technique to obtain a filter with equiripple loss in the stopband from a given all-pole function. Transmission zeros are introduced into all-pole function through a transformation without affecting the phase characteristic of the original filter.

II. THE TRANSFORMATION

Consider the transformation

\[ z^{1/2} + p z^{1/2} = \frac{z^{s/2} + z^{* \sqrt{s/2}}}{\sqrt{s}} \]  

(1)

where \( s = \cos \omega T / 2 \), \( \omega \) is the cutoff frequency of the filter in radians/s and \( T \) is the sampling interval. We shall assume \( T \) to be normalized to unity so that \( s = \cos 0.5 \). \( z \) is the complex variable of the all-pole filter function and \( p \) is the transformed variable. Letting \( p = e^{s/2} = e^{j \omega \Delta t} \) and \( z = e^{j \omega \Delta t} \), (1) reduces to

\[ \cosh (s/2) = \cosh (s/2)/\sqrt{s} \].  

(2)

For \( s = 1 \), (2) becomes

\[ \cosh (2/2) \cos (s/2) + j \sinh (s/2) \sin (s/2) = \cos (\omega t/2)/s \]  

(3)

which gives

\[ \cos (n/2) = \cos (w/2)/s \]  

(4)

for \( w, < 1 \) \( 0 < T \) in the stopband and

\[ \cos (C/2) = \cos (0/2)/s \]  

(5)

for \( 0 < 1 \) \( 0 < w \) in the passband. Thus the transformation maps the stopband in the \( z \)-plane onto the entire unit circle and the passband onto a segment of the positive real axis in the \( p \)-plane.

We use this transformation, in the next section, to obtain filters with equiripple loss in the stopband.

III. THE METHOD

Given

\[ H(z) = K \prod_{i=1}^{N/2} \frac{z^2}{z^2 + B_i z + C_i} \Delta \triangleq \frac{z^N}{h(z)} \]  

(6)

we form

\[ \hat{H}(z) = K \cdot \frac{h_m(z)}{h(z)} \]  

(7)

where \( h_m(z) \) is a mirror image polynomial (MIP) which makes the magnitude response equiripple in the stopband. Since \( h_m(z) \) is an MIP, the phase characteristic of \( \hat{H}(z) \) will be the same as that of \( H(z) \) except for a constant delay. We have assumed \( N \) to be even. For odd \( N \), the procedure is analogous.

From \( H(z) \) we obtain, using transformation (1),

\[ G(p) = \frac{1}{g(p)} = \frac{1}{\prod_{i=1}^{N} (p - p_i)} \]  

(8)

where \( p_i \)'s are the poles of \( G(p) \) and are obtained as follows. From (1), we have

\[ \frac{P_i + 1}{\sqrt{P_i}} = \frac{z_i + 1}{z_i} \]  

(9)

which gives

\[ P_i^2 + (2 - A)P_i + 1 = 0 \]  

(10)

where

\[ A = (z_i + 1)z_i \]  

(11)

and \( z_i \) is a pole of \( H(z) \). The roots of (10) give the \( p_i \)'s.

Equation (10) is an MIP and so its two roots constitute a reciprocal pair. They do not constitute a complex conjugate pair since coefficient \( (2 - A) \) is complex. However, if \( z_i \) and \( \bar{z}_i \) are complex conjugate poles of \( H(z) \), then four poles are obtained in the \( p \)-plane which occur in complex conjugate as well as reciprocal pairs. We construct \( g(p) \) using those \( p_i \)'s which lie inside the unit circle and obtain \( G(p) \) of (8). We shall call \( g(p) \) so obtained a stable polynomial, for all its roots are inside the unit circle. Now

\[ g(p) = g_s(p) + S_n(p) \]  

(12)

where \( g_s(p) \) and \( g_n(p) \) are, respectively, MIP and anti-MIP.
parts of \( g(p) \) [6]. We form

\[
\hat{G}(p) = \frac{\hat{g}_m(p)}{g(p)}
\]

so that

\[
\hat{G}(p) \hat{G}^{-1}(p) = \frac{\hat{g}_m(p)}{\hat{g}_m(p)} - \frac{\hat{g}(p)}{g(p)} = \frac{1}{1 - \left(\frac{\hat{g}_m(p)}{\hat{g}_m(p)}\right)^2}.
\]

Since \( g(p) \) is a stable polynomial, the roots of \( g(p) \) and \( g(p) \) lie on the unit circle and are interlaced [6], and \( \hat{g}(p) \) oscillates between zero and infinity for \( p = e^{j\omega} \). Therefore, \( \hat{g}(p) \) can be written as

\[
\hat{G}(p) = \frac{p^{N/2} - \ldots - \alpha_0}{\alpha_0 p^{N/2} + \ldots + \alpha_0}
\]

In the stopband \( p^{N/2} \) and in the passband \( p^{N/2} \) is a real quantity. Hence, if we drop \( p^{N/2} \) in \( \hat{g}(p) \), the stopband magnitude response and the passband phase response will remain unchanged. Dropping \( p^{N/2} \) in (15) we get

\[
\tilde{G}(p) = \frac{\hat{g}(p)}{g(p)}
\]

which is our desired filter function in the transform domain. We now go back to \( z \)-plane and obtain \( \tilde{H}(z) \) from \( \tilde{G}(p) \) through (1). \( g(p) \) is simply replaced by \( h(z) \) and \( h(z) \) is obtained from the numerator polynomial of (16) as follows.

Denoting the numerator polynomial of (16) by \( \tilde{T}(p) \), we can write

\[
\tilde{m}(p) = 2a_0 T_N(x) + 2a_1 T_{N-2}(x) + \ldots + \alpha_n
\]

where

\[
x = \frac{\cos(q/2) - \cos(q/2)}{2} = \cosh(q/2)
\]

and \( T_i(x) \) is the \( i \)-th degree Chebyshev polynomial of the first kind. From (1) and (18), we have

\[
x = y/\xi
\]

where

\[
y = \cos(s/2).
\]

Substituting \( x \) by \( y/\xi \) in (17) and expressing \( T_i(y/\xi) \) in terms of \( y/\xi \) we get

\[
\tilde{m}(p) = 2a_0 \left[ \begin{array}{c} \sum_{k=0}^{N/2} D_{N,k} \left(y/\xi\right)^k \\
\end{array} \right]
\]

where

\[
\begin{align*}
\hat{G}(p) = & 2a_0 \left[ \begin{array}{c} \sum_{k=0}^{N/2} D_{N,k} \left(y/\xi\right)^k \\
\end{array} \right] \hat{m}(z) \\
= & 2a_0 \left[ \begin{array}{c} \sum_{k=0}^{N/2} D_{N,k} \left(y/\xi\right)^k \\
\end{array} \right] \hat{m}(z)
\end{align*}
\]

Finally \( h(z) \) is obtained from \( \tilde{h}(z) \) as

\[
h(z) = \tilde{h}(z) \cdot z^N/N!
\]

Thus we have

\[
\tilde{H}(z) = \frac{\tilde{h}(z)}{h(z)}
\]

where \( k \) is a constant chosen to have unity gain of \( \tilde{H}(z) \) at \( \omega = 0 \). Since \( \tilde{G}(p) \) has equiripple magnitude response on the unit circle in the \( p \)-plane, \( \tilde{H}(z) \) will be equiripple in the stopband.

To illustrate the design procedure we take an example of an all-pole filter with the following specifications: equiripple delay over the frequency interval \( 0 \) to \( 0.4\pi \), ripple = \( k = 0.1 \) and \( N = 4 \). The transfer function of the filter is [4, Table I]

\[
H(z) = \sum_{i=1}^{2} \frac{z^2}{z^2 + B_i z + C_i}
\]

where \( B_1 = -0.057954, d = 0.33173.1, B_2 = 0.480605 \), and \( C_2 = 0.398132 \). The corresponding \( \tilde{H}(z) \) with \( 5 = \cos(e/2) = \cos 36' \) is

\[
\tilde{H}(z) = \frac{(2^2 + \alpha + 1)(2^2 + \beta + 1)}{(2^2 + B_1 z + C_1)(2^2 + B_2 z + C_2)}
\]

where \( E = 0.0531886, a = -0.4599446, \) and \( b = 1.0666315 \). The magnitude response of \( H(z) \) and \( \tilde{H}(z) \) are shown respectively in curve 1 and curve 2 of Fig. 1. The minimum stopband loss is \( 31.3 \) dB and occurs at \( \omega = 0.4\pi \).

\[ H(z) = \frac{x}{(z^2 + \beta z + 1)} \frac{(z^2 + \gamma z + 1)}{(z^2 + B_1 z + C_1)(z^2 + B_2 z + C_2)} \]

where \( \beta = 0.0212068 \), \( \alpha = 1.1106318 \), and \( \gamma = 1.8079526 \).

The magnitude response of this filter is shown in curve 3 of Fig. 1. The group delay of (29) is the same as that of (28), i.e., equiripple over the range 0 to 0.4 with ripple magnitude of 0.1 sampling interval.

V. CONCLUSION

An analytic technique of obtaining low-pass filter function with equal-ripple loss in the stopband and prescribed poles is given. The transmission zeros are introduced into the filter function through a transformation which maps the stopband of the filter onto the unit circle in the transformed plane. The technique is used to improve the loss characteristic of all-pole constant delay filters. The cutoff frequency of these filters can be chosen independent of the low-pass interval over which the delay is constant.

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REFERENCES


Some Results in Linear Interpolation Theory

STEVEN KAY

Abstract—Using a well-known form for the inverse of a symmetric Toeplitz matrix, some results in linear interpolation theory are derived. For an autoregressive process it is shown that interpolation at the midpoint of a data record yields the minimum interpolation error. Also, some results for infinite length interpolators are simply derived.

I. INTRODUCTION

Assume we are given a set of time samples of the form \( \{X-M, X-M+\ldots, Xo, \ldots, XM\} \) from a wide sense stationary random process. For simplicity we assume that the number of points \( N = 2M + 1 \) is odd. The problem is to find \( i \)