\[ A^t = \begin{bmatrix} \lambda & 0 \\ 0 & A \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 \\ 0 & A \end{bmatrix} \in \mathbb{R}^{n-1} \]  
(A.1)

and by Theorem 3.3. \( Y(A) = Y(\tilde{A}) \). Let \( c \tilde{t} \) and \( c' \) be the solutions of (2.1) and (3.1) for \( A = A' \). It follows that \( rt = [10 \ldots 0]' \), \( \tilde{t} = [1 \tilde{t}]' \), \( r = r' \).

Thus, by (A.2), \( \lambda \) and \( \bar{Y} = \gamma(\bar{A}) \) satisfy (2.3, 2.6) if the following substitutions are made: \( A = \tilde{A}, \quad r = \tilde{r}, \quad S = T\tilde{J}^r \). By the same argument used above this implies \( \bar{s} = T\bar{J}^r \) and (4.4) follows. Clearly, (4.4) implies \( e \) and \( c' \) are a basis for both \( J_{\bar{Y}} \) and \( J_{\bar{Y}} \). Thus \( A = T\tilde{A} \) shows NL. and \( J_{\bar{Y}} \). (A.4) have a common basis, which proves (4.2). Now consider part 2) ii). Note \( r, s, rt, st \) are related by \( \tilde{r} = \tilde{T}' \) and \( \tilde{s} = T'. \) Because of the orthogonality of \( T \) and the form of \( \tilde{s} \) it is not difficult to see that the right sides of (A.7) and (A.8) are linearly independent (argue this by contradiction). In turn this implies \( T^2 = T' \).

Thus \( T \) is given by (A.4). Computa-

\[ \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix} \]  
\[ = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}, \quad \bar{y}_1, \bar{y}_2 \in \mathbb{R}^{n-2}, \quad \eta > 1. \]  
(A.4)

where \( [\eta] = \lceil \frac{1}{\eta} \rceil \). Then (A.6) implies \( \bar{Y} = \gamma(\bar{A}) \). Let \( x = \gamma(\bar{A}) \) be the solutions of (2.5)-(2.7), (3.2)-(3.4) with \( r = 1 \). Thus, \( \tilde{Y}(A) = (A.6) \).

For part 2) assume \( n > 3 \). The case \( r = 1 \) is obvious from what follows. For result i) let \( r \) and \( \bar{s} \) be given by (2.5)-(2.7) and choose an orthogonal \( T \) so that its first and second columns are, respectively, \( r = 1 \) and \( \bar{s} = \bar{s} \).

Thus \( \begin{bmatrix} \eta & -\eta \omega \\ -\eta \omega & \eta \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}, \quad \bar{y}_1, \bar{y}_2 \in \mathbb{R}^{n-2}, \quad \eta > 1. \]  
(A.4)

where \( \eta = \lceil \frac{1}{\eta} \rceil \) and \( \eta = \lceil \frac{1}{\eta} \rceil \) if \( \eta < 1 \). Using \( \bar{s} \) if \( \bar{s} \) for the solutions of (2.5)-(2.7), (3.2)-(3.4) with \( A = \tilde{A} \) gives \( r = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix} \) where \( \bar{s} = \bar{s} \).

For result ii) let \( r = 1 \). Conversely, if \( \bar{s} = \bar{s} \) has the form ((4.3)) -(4.4). Theorems 4.1 and 4.2 imply the minimality of \( Y(A), \quad \gamma = \gamma(\bar{A}) \). Now consider the form of \( \gamma(\bar{A}) \). By Theorems 4.1 and 4.2 the results (4.6) and (4.7) follow immediately if \( A \) and \( W \) are replaced by \( A' \) and \( W \). Where \( W \) is defined in the obvious way from the eigenvectors of \( A' \). But the elements of \( W \) are obtained by transforming the corresponding elements of \( W \) by \( T \). The orthogonalization of \( T \) implies (4.6) and (4.7).

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I. PROBLEM STATEMENT

Consider a linear time-invariant system

\[ x(t) = (A(p)x(t) + Bu(t) \right x(0) = X^0 \]  

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(r) \in \mathbb{R}^m \) is the control vector, \( B \) is a constant matrix, and \( A \) is a matrix function of a scalar parameter \( p \) with nominal value \( A_0 \), i.e., \( A(p) = \Delta \). \( A(p) \) is assumed to be differentiable. (The extension to the case of parameter vector is straightforward.) It is assumed that

\[ \text{rank} B = m; \quad \text{rank} [B, A, B, \ldots A] = m. \]  

For a given set of distinct real eigenvalues \( \{A^1, A, \ldots, A_n\} \), there exists a control law

\[ u(t) = F(t) x(t) \]  

such that \( (A+BF)x(t) \) coincides with the spectrum of \( A+BF \). For each \( i = 1, 2, \ldots, n \), let

\[ \text{ker} [(A+BF) - A_i I] = \{ \text{eigenvectors of } A_i \} \]  

where \( Q \) and \( S \) are \( n \times n \) positive semidefinite matrices and \( R \) is a \( m \times m \) positive definite matrix and \( o(t) = \frac{\partial x}{\partial p} \) is the trajectory sensitivity vector (see Section IV). The trajectory sensitivity vector \( o(t) \) is governed by the equation

\[ o(t) = D o(t) + A o(t); \quad o(0) = 0 \]  

where

\[ A = \frac{\partial A(p)}{\partial p} \]  

The closed-loop system (1), when augmented with trajectory sensitivity equation (5), is written as

\[ \begin{bmatrix} x(t) \\ o(t) \end{bmatrix} = \begin{bmatrix} D & A \\ A & D \end{bmatrix} \begin{bmatrix} x(t) \\ o(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]  

Using (3), the performance index \( J \) given by (4) can be expressed as

\[ J = \frac{1}{2} \int_{-\infty}^{\infty} \left[ x^T \sigma' + \sigma^T x \right] \begin{bmatrix} Q + F^T FR & 0 \\ 0 & \frac{1}{3} I \end{bmatrix} \begin{bmatrix} x^T \\ \sigma \end{bmatrix} dt. \]  

Let \( F \) be given such that the closed-loop system (6) is asymptotically stable. Then the value of \( J \) associated with this selection of \( F \) is

\[ J = \frac{1}{2} \left[ x^T \sigma' + \sigma^T x \right] \begin{bmatrix} Q + F^T FR & 0 \\ 0 & \frac{1}{3} I \end{bmatrix} \begin{bmatrix} x^T \\ \sigma \end{bmatrix} \]  

where

\[ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \text{the solution of the Lyapunov equation} \]  

\[ \begin{bmatrix} D & 0 \\ A & D \end{bmatrix} P + P \begin{bmatrix} D & 0 \\ A & D \end{bmatrix} + \begin{bmatrix} Q + F^T FR & 0 \\ 0 & \frac{1}{3} I \end{bmatrix} = 0. \]  

The dependence of performance index \( J \) on initial conditions can be avoided by assuming \( E[x^T x^0] = 1 \) and minimizing

\[ E[J] = \int_{-\infty}^{\infty} \begin{bmatrix} x^T \\ \sigma \end{bmatrix} \begin{bmatrix} Q + F^T FR & 0 \\ 0 & \frac{1}{3} I \end{bmatrix} \begin{bmatrix} x^T \\ \sigma \end{bmatrix} dt. \]  

where \( E[\cdot] \) denotes the expected value [2].
Then
\[ Z := \hat{Z}^T + \hat{Z} = ZD^T + DZ \]  
\[ Z(0) = z^0 \]  
\[ (22) \]

Subtracting \( SF \) in (30), we obtain
\[ J = \int_0^\infty \Theta \left( \hat{Q} \hat{Z} \right) dt \]
\[ (23) \]

From (8b) we have
\[ \hat{b}^T \hat{P} + \hat{P} \hat{b} + \hat{Q} = 0. \]  
This gives
\[ \Theta \left( \hat{b}^T \hat{P} + \hat{P} \hat{b} + \hat{Q} \right) = -\Theta \left( \hat{Q} \hat{Z} \right). \]  
Substituting this result in (26) we get
\[ \Theta \left( \hat{P} \hat{b} + \hat{b}^T \hat{P} \right) = -\Theta \left( \hat{b} \hat{Z} \right) + \Theta \left( \hat{P} \hat{b} + \hat{b}^T \hat{P} \right) \]
\[ (27) \]

Using trace identities, we obtain the following:
\[ \Theta \left( P \right) = \Theta \left( \hat{b} \hat{Z} \right) + \Theta \left( \hat{b} \hat{P} \right) \]
\[ P \hat{b} \]  
\[ \hat{P} \hat{b} \]  
\[ \hat{b} \hat{Z} \]

From (8b) and (38) we have
\[ \Theta \left( P \right) = \Theta \left( \hat{b} \hat{Z} \right) + \Theta \left( \hat{b} \hat{P} \right) \]
\[ (28) \]

Now define
\[ \Theta \left( P \right) = \Theta \left( \hat{b} \hat{Z} \right) + \Theta \left( \hat{b} \hat{P} \right) \]
\[ (29) \]

The gradient of \( J \) with respect to unknown elements in \( F \), given by (38), is
\[ J = \sum_{i=1}^{n \times m} \left[ \Theta \left( \hat{b} \hat{Z} \right) + \Theta \left( \hat{b} \hat{P} \right) \right] \]

Then, the gradient vector is given by
\[ \frac{\partial J}{\partial a_j} = \left[ \Theta_{11} \cdots \Theta_{1m} \cdots \Theta_{n1} \cdots \Theta_{nm} \right]^T \]  
\[ (30) \]  

From (38), we obtain
\[ dF - dW_{i,j} = -dV \]  
\[ \frac{\partial J}{\partial a_j} \]

Substituting \( SF \) in (30), we obtain
\[ \Theta \left( \hat{b} \hat{Z} \right) + \Theta \left( \hat{b} \hat{P} \right) \]
\[ (31) \]

IV. THE EXISTENCE OF A SOLUTION

In the following, we prove that there exists an \( F \) that simultaneously assigns the given stable spectrum \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) to \( (A + BF) \) and

\[ (32) \]
minimizes the performance function $\bar{J}(F)$ given by (9). Let $\# = \{ F: A_0 + BF \}$ have eigenvalues $A^* A_1, \cdots, A_m$. We observe from (15) that $F(a) = F(pa)$ for any vector $a$ such that $\det K^* O$, and any nonzero scalar $p$. It is therefore sufficient to consider parameter vectors on the hypersphere

$$[a] = \left\{ I, \ldots, I, 4 \right\} \quad \forall I$$

of the nm-dimensional Euclidean space. $9$ is closed and bounded over the compact subset $\| a \| = 1$.

The performance index $\bar{J}$ is continuous in $F$ over $9$. This can be proved as follows. Equation (15) shows that each element of matrix $F$ is the ratio of two polynomials in the $nm$ elements of the parameter vector $a$. Then from (1oa), (1Oc) and (10d), it is observed that $\text{tr} P_{11}$, which is always finite for the stable closed-loop system, is the ratio of two polynomials in the $nm$ elements $a_i$ of parameter vector $a_1$. As per the Weierstrass theorem [5], $\text{tr} P_{11}$ is a continuous function on the compact set $\| a \| = 1$ and achieves a minimum on this set.

V. A NUMERICAL EXAMPLE

Consider the system

The weighting matrices are chosen as $Q = R = I$, while the eigenvalues of the closed-loop system are specified as $\lambda^i = -1, \lambda^i = -2$. With $S = 0$ in the performance index, the program was found to converge to

$$F^* = \begin{bmatrix} 0.631565 & -0.31574 \lambda^1 \\ -1.09939 & -0.117045 \lambda^1 \end{bmatrix} \quad \text{tr} F^* = 17.73$$

where the norm of the gradient was less than $10^{-6}$. With $S = 0.21$, the program was found to converge to

$$F^*_2 = \begin{bmatrix} 0.788285 & -0.96268 \lambda^1 \\ -1.37027 & 0.915121 \lambda^1 \end{bmatrix} \quad \text{tr} F^*_2 = 19.94.$$ From the simulation results of the closed-loop sl-stem $\mathcal{G} = (A, + BF^*)x$ with $x(0) = [2 \ 0]^T$, it has been observed that the off-nominal response with $p = 1.1$ is substantially degraded with respect to the nominal response with $p = 1.0$. The simulation of the system $i = (A, + BFT)x$ revealed that the dispersion of the off-nominal response from the nominal one is smaller compared to the earlier case.

VI. CONCLUSIONS

The freedom of the choice of the feedback gains that assign a desired spectrum to the closed-loop system is utilized to minimize a given performance index which is of standard linear regulator type. Modified to include a quadratic sensitivity term. The results have been derived for the case of distinct eigenvalues. However, generalized results for the case of repeated eigenvalues can easily be obtained [6].

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On the Robustness of LQ Regulators

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Abstract—It is shown that in spite of its large gain and phase margins the linear quadratic state feedback regulator may suffer from poor robustness where small changes in the parameters of the system lead to fast unstable closed-loop modes.

INTRODUCTION

One of the attractive features of the multivariable linear quadratic optimal regulator is its guaranteed classical gain margins of $-60$ to $+x$ dB and phase margins of $+60$ deg in all channels [1]-[3]. This property has led to the assertion that the optimal linear quadratic regulator should possess good robustness properties, whereas its related LQG controller-filter design suffers from poor robustness properties to the extent that arbitrarily small changes in the parameters of the open-loop plant may cause instability [4], [5].

It has also been shown in [3] that in the special case of nondiagonal weighting matrix on the control and a special input matrix, a small multiplicative change in the latter matrix may cause instability even in the linear quadratic regulator case. Th-s stems. however, from an effective large change in the normalized input matrix and not necessarily from an inherent deficiency of the controller.

It is demonstrated by the following example that in spite of its impressive margins the full state linear quadratic optimal regulator may suffer from robustness problems where small changes in the parameters of the system may lead to fast unstable closed-loop modes. This poor robustness is encountered also for small changes in the normalized input matrix and it does not depend on the diagonal(1); of the control weighting matrix.

EXAMPLE

We consider the well-known linear quadratic state feedback optimal regulator problem [1] for the single input, single output linear time-invariant continuous system $S(A, b, c)$ given by

$$x(t) = Ax(t) + Bu(t) \quad x(0) = x_0 \neq 0$$

$$y(t) = c^T x(t)$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 1 \end{bmatrix}.$$ A state feedback control should be found that minimizes the following index of performance:

$$J = \rho \left[ y(t) + \rho u(t) \right] dt, \quad \rho > 0.$$