DESIGN OF hR FILTERS USING PSEUDO-BOOLEAN METHODS

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Abstract
Pseudo-Boolean methods are particularly suited to problems in which the variables take only two values, 0 and 1. In this paper, a method is presented for solving nonlinear pseudo-Boolean inequalities, and is applied to limited word length design of hR digital filters. Starting with an approximate solution, the proposed design method adjusts the quantized coefficients to achieve the best filter characteristics. It takes much less computation time as compared to the existing search techniques and overcomes the problem of global optimum, inherent in them techniques. The use of the method is demonstrated through an example of a third order Chebyshev filter.

INTRODUCTION
In most of the digital filter design methods, the filter parameters are assumed to be specified to an infinite accuracy. To implement them on a digital machine, these parameters have to be quantized to a prespecified number of binary digits. This results in a deterioration of the filter characteristics, the extent of which depends upon the type of quantization employed. In general, the simplest of these, viz. rounding or truncation does not lead to the best filter characteristics obtainable with the specified number of bits. Several methods [1]-[4] have been suggested with a view to improving the characteristic by adjusting the rounded or truncated values of the design parameters. Based on nonlinear search techniques in a discrete parameter space, these methods do not always converge to a global optimum and take a large amount of computation time. In this paper, these two problems are overcome, to a large extent, by a simple technique based on pseudo-Boolean methods, consisting of a combination of dynamic programming and Boolean techniques. The use of linear pseudo-Boolean methods in designing FIR filters has been demonstrated elsewhere [5], here we consider hR filter design using modified non-linear pseudo-Boolean techniques.

PSEUDO-BOOLEAN METHODS
In this Section, we briefly discuss the basic concepts underlying pseudo-Boolean methods [6]-[8]. A two element Boolean algebra, B2 consists of the operations of disjunction (U), conjunction (.) and negation on the elements of the set 0, 1. A pseudo-Boolean (PB) function is, by definition, a real valued function of bivalent variables, each variable taking one of the two values 0 and 1, and can always be written in the form of an arithmetic expression because, with x, y E {0,1}, x+y = x + y -xy ; x. y = x . y = 1 - x.

In the arithmetic form, a linear PB inequality, in its general sense, can be written as follows:
\[ \bigvee_{i=1}^{n} a_i x_i + b \geq k \]
where a_i, b, \( i + 1 \leq n \) and k are real values and \( x_i \)'s are bivalent variables. The other forms of inequalities (viz. =, >, <) can easily be converted to this form. For solving (1), it is first transformed into the canonical form [6],[1]
\[ c_1 x_1 + \ldots + c_n x_n \geq d \]
such that c_i's and d are real values ; x_i's t. \{0,1\}, and the terms are indexed such that c_1 \geq c_2 \geq \ldots \geq c_n 0. There are many solutions to (2), which can be grouped into families, a family being defined as one in which some of the variables have fixed values and the others take arbitrary (either 0 or 1) values. These families of solutions can be obtained by systematic use of a set of rules given in [7]. As an example, if \( d < c_1 \), then there exists no solution, i.e., if \( d < c_1 \), the inequality is redundant i.e. all the variables can take arbitrary values.

Solutions of a system of linear PB inequalities can be tracked by following a similar set of rules ; one inequality is considered at a time and a conclusion regarding some of the variables is drawn using these rules. This leads to another conclusion referring to the whole system [1].

In the context of hR filter design, one comes across nonlinear PB inequalities. The solutions are obtained by converting them to the linear form, based on the fact that if \( x, y, \ldots 1 \geq 1 \), then \( x \) is also a bivalent variable. This method, however, increases the number of variables and, therefore, takes a larger computation time.

In the next Section, an alternate method, particularly suited to hR filter design, is described.

ALTERNATE PB METHOD FOR hR FILTERS
The method will first be described in general terms and then reduced to the special case of hR filters. Considers nonlinear PB inequality
\[ f(x_1, \ldots x_n) \geq u \]

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According to a PB theorem [6], f can be made linear in any one of its variables. If \( x_i \) is that variable then (3) can be written as

\[
x_i \leq \frac{1}{2} (x_1 + \ldots + x_n) - \frac{1}{2} \sum_{j=1}^{n} y_j \tag{4}
\]

which can be solved for \( x_i \) using linear PB methods if it can be converted to a canonical form; the required steps for the solution are shown in the flow graph of Fig. 1. For conversion to the canonical form, it is necessary to determine the values of the variables \( x_1, \ldots, x_n \) for which \( g_i \geq 0 \) or \( < 0 \). This can be done by solving PB inequality \( g_i \geq 0 \), which is similar to (3) and can be written as a linear inequality in \( x_i \), i.e. \( x_i \geq \frac{b_i}{a_i} \), where \( g_i, h_i \) are constants.

The method becomes very simple when applied to the case of \( h \times r \) filter design. To illustrate this, consider a second order transfer function

\[
H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2}
\]

The \( a_i \)'s in (5) can be obtained, to an infinite accuracy, by using any of the conventional design methods [9]. Let \( b_i \) be the values of \( a_i \) rounded to \( n \) bits with \( q \geq 1 \). The resulting deteriorated magnitude response of (13) is shown by the curve (a) in Fig. 2. In the same figure, the magnitude response marked (c) is the one with coefficients rounded using the method presented in the last section. These coefficients are

\[
a_0 = 0.01195621775, \quad a_1 = -1.9744602360, \quad a_2 = 1.5242778380
\]

and

\[
c_0 = q(q + 2b + 2b_1 \cos \theta_j) + \ldots
\]

where, for \( j = 1 \) to \( m \),

\[
c \leq \frac{q(q + 2b + 2b_1 \cos \theta_j)}{2}
\]

Further, note that \( q = r \), and \( s \) are linear PB functions so that \( g_i > 0 \) and \( g_i < 0 \) can be solved simply \( g \) using linear PB methods. Solution of these can also be avoided if \( g_i < 0 \), or both are either always positive or \( g \) This can be done quite often by proper choice of the variables with respect to which (8) is made linear.

The method will give many solutions and the one giving the best frequency response is chosen. The method is repeated after modifying the values of \( a_0, a_1, a_2 \) and \( f \). i.e. 5, is replaced by \( -5 \) if \( y_i = 0 \). The computation process ends when \( n \) further improvement in the response can be obtained.

**EXAMPLE**

The method has been used to improve the frequency response of a third order Chebyshev filter ([9], p. 74) with cutoff at 0.1944 (35°) and \( b = 7 \). The filter coefficients are \( \frac{1}{2} \) and \( -0.4537678600 \). The magnitude squared response of (13) is shown by the curve (a) in Fig. 2. If the coefficients are rounded to seven bits after decimal, then the coefficients become

\[
s_0 = 0.015625, \quad s_1 = -1.96875
\]

and

\[
a_0 = 0.190625, \quad a_1 = -1.96875
\]

The simplicity of the proposed method in (8) arises due to various reasons. First, note that (8) can be made linear simultaneously in two variables \( r_1 \) and \( r_2 \) which belong to the numerator and \( f \) or the denominator. Linear PB methods can be used more effectively if this can be done. Made linear in \( y_i \) and \( y_i \), (8) looks like

\[
V_i = \frac{3V_3}{2} - \frac{1}{2} (V_1 + V_2) + \frac{1}{2} V_3
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where

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V_1 = c_0^2 + c_0^2 + c_2
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Formulate new ineq. \( x_1 q_1 \geq h \)

Fig. 1 Flow chart for tracking the solutions of the PB inequality \( x_1 q_1 \geq h \).
Fig. 2 Frequency response of the hR filter
with a) coefficients not quantized,
b) coefficients rounded to seven bits
after decimal, and c) modified quantized coefficients, using PB methods.