Single-Mode Optical Waveguides and Directional Couplers with Rectangular Cross Section: A Simple and Accurate Method of Analysis

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Abstract—A new method for obtaining the propagation characteristics of single-mode optical waveguides with rectangular cross section is presented. The method is based on the scalar variational principle using the cosine-exponential trial field. This form of trial field leads to the definition of equivalent guiding structures consisting of homogeneous slab waveguides, which are then used to obtain, in a very simple way, the vector modes of a rectangular waveguide and the coupling characteristics of directional couplers consisting of two rectangular waveguides. We have also included the results of some representative calculations along with the results obtained using other approximate methods and the exact numerical method. A comparison shows that our method is much more accurate than the existing approximate methods, particularly in the region of single-mode operation.

I. INTRODUCTION

SINGLE-MODE optical waveguides with rectangular cross sections are the building blocks of most integrated optical devices; hence, a knowledge of their propagation characteristics is important for the design of such devices. The evaluation of the propagation characteristics involves solving the electromagnetic boundary value problem for the dielectric guiding structure. However, it is not possible to obtain analytical solutions of the boundary value problem for rectangular waveguides; one has to either use numerical techniques [1]-[8] or develop approximate solutions [9]-[15]. The numerical techniques, such as circular harmonic analysis [1], direct numerical integration of the field equations [2]-[3], finite element analysis [4], field expansion in orthogonal functions [5]-[8], etc., require extensive computations and do not lead to simple analytical forms for the modal fields. This has led to the development of various approximate methods.

The earliest approximate method is due to Marcatili [9], and it gives good accuracy for a well-confined mode. However, most integrated optical devices are based on single-mode waveguides, in which the propagating mode has a substantial fraction of its power outside the channel region. Hence, Marcatili’s analysis leads to considerable error for such practical waveguides. This approximation was improved upon by Knox et al. [10] by introducing the concept of “effective index.” Although the effective index method provides a better approximation and has generally been used [10]-[12], the accuracy that can be obtained by this method for single-mode waveguides is only marginally better than that of Marcatili’s method. Perturbation [13], [14] or variational [15] methods can be used to improve the accuracy of the propagation constant evaluation to some extent; however, these do not improve the accuracy of the modal field calculation.

We propose here a simple approximation for the propagation characteristics of single-mode rectangular waveguides and show that it is extremely accurate in the practical region of single-mode operation. The method is based on the scalar variational analysis with a new and more accurate trial field [16]. This trial field enables us to define equivalent waveguiding structures consisting of slab waveguides. Using the equivalent guiding structures, we have obtained, in a simple way, the two fundamental vector modes $E_x$ and $E_y$ of the original rectangular waveguide [17]. A comparison with available exact numerical results shows that the results obtained using the present method are much more accurate than those obtained using other approximations; hence, the present approximation is more suitable for evaluating characteristics of single mode waveguides. Furthermore, the equivalent guiding structures obtained in our method also greatly simplify the analysis of composite guiding structures such as directional couplers [17].

II. THE SCALAR MODE

As mentioned above, our method consists of first obtaining a simple and accurate model for the modal field in the scalar approximation and then using this model to define equivalent guiding structures, which allow accurate approximation of the vector modes, $E_x$ and $E_y$. In this section we discuss the model for the scalar mode.

We consider a waveguide with a refractive index dis-
stationary expression for $\lambda$ (see, e.g., [18]):

$$n(x, y) = n_c, \quad y > b, \quad \text{all } x$$

$$= K, \quad |y| < b, \quad |x| < a$$

$$= n^2_{\rho}, \quad |y| < b, \quad |x| > a$$

$$= n^2_s, \quad y < -b, \quad \text{all } x$$

(1)

where all dimensions are defined in Fig. 1(a). Such a structure reduces to a rectangular waveguide when $n_p = n_s = n_c$ to a channel waveguide when $n_s < n_p = n_c < n_o$, and to an embossed waveguide when $n_s = n_p < n_c < n_o$.

A suitable model for the scalar mode can be obtained by assuming a modal field distribution of appropriate form involving adjustable parameters and using this field in the variational analysis for the propagation constant ($\beta$). The values of the parameters that maximize ($\beta$) define a field which closely approximates the actual modal field, and the value of ($\beta$) thus obtained is also very close to the actual propagation constant [18]. The accuracy of the modal field and the propagation constant ($\beta$) thus obtained depends on the choice of the trial field. The trial field can be either a single function or a series of orthogonal functions. The latter class of trial fields require extensive computations and are more suitable for use with multimode waveguides [5]-[8]. A variety of single function trial fields have been suggested and used [16], [19]—[21], but of these the cosine-exponential trial field are given in [16]. The field is assumed to be separable in its $x$- and $y$-dependences:

$$\psi(x, y) = \psi(x, y) \psi(y)$$

(2)

and $\psi_x$ and $\psi_y$ are given by

$$\psi_x(x) = A_x \cos (px/a), \quad x \leq a$$

$$= A_x \cos (px/a) \exp [-i/2\{1 - 1 \tan (p/a)\}], \quad x > a$$

(3)

and

$$\psi_x(y) = A_2 \cos [qy/b - a], \quad y \leq b$$

$$= A_2 \cos (q - a) \exp [q(y/b - 1)] \tan (q - a), \quad y > b$$

$$= A_2 \cos (q + a) \exp [q(y/b + 1)] \tan (q + a), \quad y < -b$$

(4)

where $p$, $q$, and $a$ are the three adjustable parameters, suitable values of which are obtained by maximizing the stationary expression for $\beta$ (see, e.g., [18]):

$$\beta = \frac{k_0^2}{\kappa} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla \psi|^2 dx dy$$

where $k_0 = \omega/c$ is the free-space wavenumber. Substitution for $^\psi(\psi, \psi)$ from (2)-(4) in (5) and straightforward integration result in a closed form expression for $\beta^2$ in terms of $p$, $q$, and $a$ [16]. The variation of $\beta^2$ is then maximized with respect to $p$, $q$, and $a$ using a standard optimization routine. The main advantage of this form of the trial field lies in the fact that both $V_x(x)$ and $V_x(y)$ correspond to the exact modal field of some slab waveguide which can be defined once $p$, $q$, and $a$ are known. This allows us to define an equivalent guiding structure for the rectangular waveguide, as discussed in Section III.

In order to illustrate the accuracy of the cosine-exponential trial field, we have included here a typical example. We consider a rectangular waveguide ($n_s = n_p = n_c$) with $n_o - n_s = n_s$ with an aspect ratio of $b/a = 0.5$. The variation of the normalized propagation constant

$$\nu = \frac{\beta k_0}{n_o^2 - n^2_s}$$

(6)

as a function of the normalized frequency

$$V = k_0 a (n_o - n_s)$$

(7)

is shown in Fig. 2 along with the results obtained by using other approximate methods and the exact numerical results [1]. More examples to illustrate the accuracy of using the cosine-exponential trial field are given in [16].

III. THE EQUIVALENT GUIDING STRUCTURES (EGS)

As mentioned in Section II, the forms of $V_x(x)$ and $V_x(y)$ correspond to the exact scalar modes of step-index slab waveguides with index variation along the $x$-axis and the $y$-axis, respectively. Thus, one can define a slab...
The determination of $H_0$ and $n_x$ poses some problem, which can be solved in the following way.

Since $p$ and $q$ represent the transverse phase constants along the $x$- and the $y$-directions, respectively, and $\theta$ is the phase constant along the $z$-direction, these must satisfy the relation

$$p^2 + q^2 \ell^2/b_0^2 + (3^2 \ell^2 = (k_o x, y)^2$$

within the accuracy of the variational method. By analogy, we can define $n_x$ such that

$$(k_o a, b) = p^2 + (3^2 \ell^2 = (k_o a, b)$$

It may be noted that as the dimension of the waveguide along the $y$-axis increases, $qa/b$ decreases, approaching zero as the dimension tends to infinity (i.e., when the rectangular waveguide reduces to a slab waveguide), and $n_x$ approaches $n_o$ which should indeed be the case. Thus, $n_x$ from (13) and $n_x$ from (9) define a slab waveguide with dielectric interface in the $y-z$ plane; we shall refer to this as the $x$-slab. This $x$-slab is such that the propagation constant $3$ and the phase constant along the $x$-axis $p$, for its mode are the same as those for the mode of the actual waveguide (within the accuracy of the variational method). Therefore, this $x$-slab is equivalent to the given waveguide as far as the characteristics along the $x$-direction, i.e., those characteristics which have to take into account the $y-z$ interface, are concerned. For example, for obtaining the coupling characteristics of a directional coupler with its constituent waveguides separated along the $x$-axis (see Section V) or for obtaining the vector mode coupler, are concerned. For example, for obtaining the coupling characteristics of a directional coupler with its constituent waveguides separated along the $x$-axis (see Section V) or for obtaining the vector mode

$$V_H \left( \frac{\partial}{\partial z} \right) = 0,$$

and the corresponding wave equation would be

$$d^2 \psi/dx^2 + \left( k_o n(x) \right)^2 \psi = 0.$$
and the corresponding wave equation would be

\[ \frac{d^2\psi}{dy^2} + [kW(y) - (\beta^2)] \psi(y) = 0. \quad (18) \]

We shall make use of these equivalent guiding structures (EGS) to obtain the vector modes of a rectangular waveguide in Section IV and to obtain the coupling characteristics of directional couplers consisting of two parallel rectangular waveguides in Section V.

IV. THE VECTOR MODES

In general, a rectangular waveguide which supports a single scalar mode would allow two orthogonally polarized vector modes, namely \( E_x \) and \( E_y \), [1], [9], which have slightly different propagation properties. These two modes, if coupled by unavoidable irregularities, may cause undesirable effects when a polarization sensitive device is used at the end of the waveguide. Thus, in order to design efficient integrated optical devices, such as directional couplers, filters, mode converters, etc., it is of interest to study the vector modal properties of such rectangular-core waveguides. There are few methods available for this purpose. Of these the circular harmonic analysis by Goell [1] requires extensive numerical computations, and the closed form solution given by Marcatili [9] essentially neglects the modal field in the corner regions, thereby resulting in poor accuracy. We make use of the EGS discussed in Section III to obtain the vector modes of a rectangular core waveguide.

The two fundamental vector modes \( E_0 \) and \( H_y \) are hybrid in nature and all the components of the electric and the magnetic fields are nonzero. However, the \( E_0 \) mode has strong \( E_x \) and \( H_y \) components, with its other components described by

\[ E_0, H_y \gg E_x, H_x. \]

Hence, it behaves as a TM \( \beta \) mode in the x-slab (of EGS) and as a TE \( \beta \) mode in the y-slab. Thus one can approximate \( E_0 \) by

\[ E_0 = F_x(x) F_y(y) \quad (19) \]

where \( F_x(x) \) corresponds to the TM \( \beta_0 \) mode of the x-slab and \( F_y(y) \) corresponds to the TE \( \beta_0 \) mode of the y-slab. Hence

\[ F_A(y) = \%(y) \quad (20) \]

and

\[ \text{and } F_A(x) = \langle \psi^2_r \rangle \cos (p_a x), \quad \text{and } \quad \langle \psi^2_r \rangle \cos (p_a x), \quad \text{and } \quad \langle \psi^2_r \rangle \cos (p_a x) \]

where \( a^2 = p^2 \sec^2 (p^2/2p^2) \) and \( \beta_0 \) is obtained by solving the eigenvalue equation corresponding to the TM \( \beta_0 \) mode of the x-slab; that is

\[ \beta_0 = a(n_2/\sqrt{n_0}) \] \( \beta_0 = a(n_2/\sqrt{n_0}) \] 

The propagation constant, \( i_n \), of the \( E_{n\ell} \) mode is thus obtained from

\[ \beta_n^2 = \beta_0^2 + (p^2 - \beta_0^2)/a^2 \quad (23) \]

where \( \beta_0 \) is the scalar propagation constant. For the \( L_e^{\ell\ell} \) mode, one can carry out a similar analysis by finding the eigenvalue corresponding to TM \( \beta \) mode of the y-slab.

To illustrate the accuracy of the method, we consider a rectangular waveguide with \( n_0 = 1.5, n_1 = n_2 = 1.0 \) and aspect ratio \( b/a = 0.5 \), for which the exact results are available [1]. The variation of the normalized propagation constant \( B \) as a function of \( K/TT \) for the two fundamental vector modes is shown in Fig. 3, along with the results obtained by using Marcatili’s method [9] and the exact numerical results of Goell [1]. As can be seen from the figure, our approximation gives an extremely accurate description of the fundamental vector modes in the region of interest for the operation of integrated optical devices, i.e., in the wavelength region just below the cutoff of the next higher mode, which is about \( V = 0.8TT \) [1].

V. DIRECTIONAL COUPLERS

A directional coupler consists of a pair of waveguides placed parallel to each other so that the evanescent field of one waveguide penetrates into the other waveguide and couples into the latter waveguide’s propagating mode. If one of the two waveguides is excited, then after a certain length, most of its power is transferred to the other waveguide. The length of the coupler at which maximum power transfer occurs is referred to as the coupling length \( L_c \). The coupling length depends upon the structure and refractive indices of the constituent waveguides and the intervening isolation region. These directional couplers are used in a variety of integrated optical devices such as power dividers, input-output couplers, modulators, filters, switches, etc.

In this section we again use EGS and obtain the coupling characteristics of a directional coupler consisting of two identical rectangular-core waveguides (see Fig. 4). Since the two waveguides are separated along the x-axis, we use the equivalent x-slab waveguides. The refractive index distribution of the equivalent slab coupling structure is given by (see Fig. 4)

\[ n_2(x) = n_2, \quad \text{and } \quad n_2, \quad \text{and } \quad n_2 \]

\[ \text{where } 2d \text{ is the separation between the two waveguides and } n_1, n_2 \text{ are defined in Section III. Thus, the problem of coupling between two rectangular waveguides reduces to a much simpler problem of coupling between two slab waveguides, and one obtains an analytical expression for the coupling length } L_c \text{ (see, e.g., [23])} \]

\[ U_{-1} \propto T[1 + \tan (\beta /2)] \exp \{\beta (2d/a)/2 \}

\[ (25) \]
where $\beta$ and $\rho$ correspond to each of the two constituent waveguides and are obtained from the variational analysis as discussed in Section II. In Fig. 5, we have shown the variation of the normalized coupling length $L_c/\lambda a^2$ as a function of the normalized separation $d/a$ for a typical directional coupler consisting of two identical rectangular waveguides. The corresponding calculations using Marcatili's method [9] lead to considerably different results for the coupling length, as shown in Fig. 5. The difference is expected due to the inherent error in Marcatili's method, as shown in Fig. 2.

VI. SUMMARY AND CONCLUSION
We have developed a new method to obtain the characteristics of homogeneous waveguides in rectangular cross sections. The method, which is based on the scalar variational calculations, defines for the given waveguide equivalent guiding structures which can then be used to
obtain the vector modes of the waveguide and the coupling characteristics of a directional coupler made up of two such waveguides. The accuracy of the method and of the calculation of vector modes is illustrated with examples, and it is shown that this approach gives much more accurate results than those obtained using other approximate methods.

REFERENCES


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