the system fibre would therefore permit relatively large separations between repeater/optical pump connections.

D. M. SPIRIT
I. W. MARSHALL
P. D. CONSTANTINE
D. L. WILLIAMS
S. T. DAVEY
B. J. AINSLIE

British Telecom Research Laboratories
Martlesham Heath
Ipswich IP5 7RE, United Kingdom

References


**RECURSIVE CONSTRUCTION OF POLYNOMIAL WITH PRESCRIBED PHASE ON UNIT CIRCLE**

**Indexing terms:** Computer software, Fourier transforms

An extension of a recent technique for phase interpolation to make it implementable on a computer is described. A computer program has been written and several examples are presented to demonstrate its usage.

**Introduction:** It has been well recognised in the literature¹ that many features of a signal are preserved if only the phase is retained. It has also been shown² that under certain conditions, a signal can be uniquely reconstructed to within a scale factor from the phase or samples of the phase of its Fourier transform.

Several algorithms³ have been developed for this purpose. In a recent contribution, Merchant¹ proposed a recursive technique for constructing a polynomial, \( P(n) \), of order \( N \), which satisfies phases at \( N \) given frequencies. His method, however, is not directly programmable. Here, we extend this technique for computer implementation. The necessary program* has been written in Fortran; several illustrative examples are presented to demonstrate the accuracy and efficiency of the program.

The problem: Let \( PV(n) \) be the \( z \)-transform of a discrete-time finite duration signal \( p(n), 0 \leq n \leq N \), i.e.

\[
P(z) = \sum_{n=0}^{N} p(n) z^{-n}
\]

The Fourier transform of the signal is simply obtained by replacing \( z \) by \( e^{j\theta} \) in eqn. 1. Let

\[
P(e^{j\theta}) = \sum_{n=0}^{N} p(n) e^{-j\theta n}
\]

and let \( \theta = \{\theta_1, \theta_2, \ldots, \theta_N\} \) be specified at \( N \) distinct frequencies: \( \theta_k = k\pi/N \), \( k = 1, 2, \ldots, N \), where \( 0 < \theta_k < \pi \) and \( \sin \theta_k \) need not be ordered or uniformly spaced. It is assumed that the phase values do not contain a linear phase component. The problem is to reconstruct the signal from the given frequency-phase pairs \( (w_k, \theta_k) \).

Merchant's solution: Merchant¹ viewed this problem as that of determining a sequence of monic real polynomials \( P\left( e^{j\theta} \right) \), of order \( m \), \( m = 2, 3, \ldots, N \), satisfying the given frequency-phase characteristics on the unit circle, i.e.

\[
\text{phase} \left[ P\left( e^{j\theta} \right) \right] = \theta \quad \text{for} \quad 0 \leq \theta \leq \pi
\]

He showed that \( P\left( e^{j\theta} \right) \) is related to its predecessors \( P\left( e^{j\theta} \right) \) and \( P\left( e^{j\theta} \right) \) as follows:

\[
P\left( e^{j\theta} \right) = \sum_{n=0}^{\infty} p_n e^{-j\theta n} \quad \text{with} \quad p_0 = 1
\]

\[
p_1(\theta) = 1
\]

\[
p_1(\theta) = -\sin \theta
\]

\[
\frac{\sin \theta_1}{\sin \theta_1 + \sin \theta_2}
\]

Note that if \( \sin \theta_1 + \sin \theta_2 = mn \), \( n = 0 \) or any integer, then \( p_1(\theta) \) cannot be computed. In such a case, \( (\theta_1, 0) \) should be interchanged with some other \( (\theta_1, \theta_2) \), such that \( \sin \theta_1 + \sin \theta_2 = nn \).

Extension: Merchant's recursive formulation, though very elegant, is not programmable on a computer. To facilitate this, we substitute in eqn. 4 the corresponding polynomial expressions for \( P\left( e^{j\theta} \right) \) and \( P\left( e^{j\theta} \right) \) and carry out somewhat lengthy algebraic manipulations. This results in the following recursive formulas for the coefficients of the polynomial:

\[
p_m(\theta) = \sum_{r=0}^{m} p_{m-r}(-1)^r \left( \cos \theta_{m-r} - \cos \theta_{m+r} \right)
\]

where

\[
p_m(\theta) = 0 \quad \text{for} \quad j > i
\]

* May be obtained from the authors on request

1. A monic polynomial is defined as one in which the coefficient of the highest power term is unity.
The parameters $\beta_{-i}$ and $\alpha_{-i}$ in eqn. 6 are given by

$$\beta_{-i} = \frac{a_{-i}}{(pM-D) + p^2 \sin(i\gamma) + 2B_{wa} \cos(\gamma \alpha) - \cos \gamma \alpha}$$

(7a)

$$\gamma = 1, \ldots, \gamma_{-i} = \frac{D - p^2 \sin \gamma (D)}{2} \gamma_{-i}$$

(7b)

where

$$a_{-i} = \sin \gamma \gamma_{-i} - \frac{1}{2} \sin(i\gamma) \sin \gamma \gamma_{-i} + \frac{1}{4} \sin(i\gamma) \sin \gamma \gamma_{-i} + \frac{1}{8} \sin(i\gamma) \sin \gamma \gamma_{-i}$$

(7c)

and

$$a_{-i} = \frac{1}{2} \sin(i\gamma) \sin \gamma \gamma_{-i} + \frac{1}{4} \sin(i\gamma) \sin \gamma \gamma_{-i} + \frac{1}{8} \sin(i\gamma) \sin \gamma \gamma_{-i}$$

(7d)

Merchant\(^1\) has cited two situations either of which will cause the recursion to terminate, and has also suggested remedial actions; these have been incorporated in the program. However, it may happen that even after taking care of these situations, the recursion may terminate at a polynomial of order less than the desired order $N^2$. This indicates the presence of a linear phase component in the full set of frequency-phase pairs which needs to be removed before the polynomial can be constructed. With premature termination, our program reports the last polynomial successfully constructed. As an example, consider the polynomial

$$P(z) = (1 + z^{-1} + z^{-2} + z^{-3})$$

Due to presence of a symmetric factor in $P(z)$, recursion cannot proceed beyond the computation of $P(z)$ (an intermediate polynomial). However, it was observed that for

$$P(z) = (1 + z^{-1} + z^{-2} + z^{-3})$$

and

$$P(z) = (1 - z^{-1} + z^{-2} + z^{-3})$$

the presence of linear phase factors does not terminate the recursion and $P(z)$ and $P(z)$ are successfully constructed. It is inferred that zeros at $z = 1$, though contributing to linear phase components, do not hinder the unique construction of a sequence from its phase values.

To further validate the program, we consider a polynomial of order 6

$$P(z) = 1 + 2z^{-4} + 3z^{-5} + z^{-6} + 4z^{-7} + 5z^{-8} + 6z^{-9}$$

(8)

The phase values at an arbitrary set of frequencies are listed in Table 1. The coefficients obtained by implementing the procedure given here on an HP 9000 system are also shown in Table 1. The error in computed values, though negligible, is attributed to the finite-bit arithmetic of the computer.

S. MINOCHA

27th November 1990

B. KUMAR

Department of Electrical Engineering

Indian Institute of Technology, Delhi

Hauz Khas, New Delhi—110 016, India

REFERENCES


### Table 1. COMPARISON OF THE COMPUTED VALUES OF THE COEFFICIENTS WITH THEIR ACTUAL VALUES FOR THE EXAMPLE (Eqn. 8)

<table>
<thead>
<tr>
<th>$\omega$ (radians)</th>
<th>$\omega$ (radians)</th>
<th>$\omega$ (radians)</th>
<th>$\omega$ (radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.8599681490555361</td>
<td>2</td>
<td>1.999999998171184</td>
</tr>
<tr>
<td>0.9</td>
<td>1.084476849144754</td>
<td>3</td>
<td>3.000000002299421</td>
</tr>
<tr>
<td>0.71/10</td>
<td>1.084476849144754</td>
<td>4</td>
<td>0.99999999818273</td>
</tr>
<tr>
<td>0.2/5</td>
<td>4.5561205210889 x 10^{-2}</td>
<td>5</td>
<td>4.000000002017118</td>
</tr>
<tr>
<td>0.3/7</td>
<td>0.546725253692545</td>
<td>6</td>
<td>4.99999999922391</td>
</tr>
</tbody>
</table>

Owing to monic polynomial condition, $p(0) = 1$.