An Optimum Detector for Coherent M-ary Signaling in The Presence of Impulsive Noise

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Abstract — Digital communication systems operating in the ELF/VLF range are strongly affected by atmospheric noise which has an impulsive component on a white Gaussian background. We derive an optimum detector for coherent M-ary signaling in the presence of such noise. In the special case of coherent binary signaling, the performance of a suboptimum linear detector is analyzed.

I. INTRODUCTION

Digital communication systems in the ELF/VLF range operating in situations like a mine environment are strongly affected by atmospheric noise which is impulsive in nature. Various models for such noise, such as a shot noise model [1], a narrowband model with a lognormal envelope [2], and an alpha-stable process model [3] have been considered. The performance of coherent and noncoherent detectors using such models has been analyzed [4] [5] [6]. An optimum receiver for on-off signaling in the presence of sampled impulsive noise with a continuous probability density function (p.d.f.) has been derived in [7].

In this paper, we model atmospheric noise in the ELF/VLF communication environment as the sum of an impulsive component and a Gaussian component, as in [1] [2] [4] [11]. Since the impulsive component arises from natural phenomena like lightning discharges, it is appropriate to model it as a sum of random-amplitude narrow pulses with Poisson arrivals. Using the maximum a posteriori probability (MAP) criterion, an optimum detector for coherent M-ary signaling in the presence of such noise is derived.

II. THE RECEIVED SIGNAL MODEL

Consider a coherent M-ary communication system with real signaling waveforms \( s_0(t), \ldots, s_{M-1}(t) \) supported over the signaling interval \([0,T_b)\). Denoting the hypothesis "\( s_i(t) \) has been transmitted" as \( H_i \), \( i = 0, \ldots, M-1 \), the real received signal \( r(t) \) under \( H_i \) is modeled as

\[
H_i : r(t) = s_i(t) + n(t), \quad 0 \leq t < 2T_b.
\]

where waveform \( s_i(t) \) is transmitted for symbol \( i, i = 0, \ldots, M-1 \), \( T_b \) is the bit duration, and \( n(t) \) is the total noise. The noise is considered to be the sum of two independent random processes, an impulsive component \( v(t) \) and a zero-mean white Gaussian component \( w(t) \). Thus

\[
n(t) = v(t) + w(t),
\]

in which

\[
v(t) = \sum_{m=1}^{\infty} a_m q(t - T_m),
\]

is a sum of narrow pulses with random delays \( \{T_m\} \) and random amplitudes \( \{a_m\} \), each having p.d.f. \( f_a(.) \). Note that \( \{a_m\} \) and \( \{T_m\} \) are independent random sequences. The pulse \( q(t) \) satisfies

\[
\int_{0}^{\infty} q(t - T_m) q(t - T_n) dt = 0 \quad \text{if} \quad T_m \neq T_n \quad \text{and} \quad E[v(t)] = E[w(t)] = 0
\]

\[
E[v(t) - T_m] p(t) dt = A \delta(t - T_m)\]

for all \( r_m T_b \in [0, T_b) \), and any function \( p(.) \). In (3) \( \delta(t) \) denotes the pulse energy, and \( A, (A > 0) \) the pulse area. The arrival of the pulses is modeled as a Poisson process. The performance of the detector will be analyzed in Section III.

The zero-mean white Gaussian noise component \( w(t) \) has two-sided power spectral density \( U_w \); that is,

\[
S(w(h)w(t)) = 2U_w \delta(t - t')
\]

\( \mathcal{E} \) being the expectation operator.

Let \( v(t,K(T_b)) \) denote the impulsive component \( v(t) \) given that \( K(T_b) \) pulses arrive in the interval \([0,T_b)\). Then we have

\[
v(t,K(T_b)) = \sum_{\tau \in \mathcal{K}} a_{\tau} q(t - \tau),
\]

where \( \tau_l, l = 1, \ldots, K \) are i.i.d. random variables uniformly distributed over the interval \([0,T_b)\) owing to Poisson arrivals of the pulses.
Consider a sequence \( \{ f_i(t) \} \) of \( M \) orthonormal functions on the signaling interval \( [0, T_b) \) satisfying
\[
J_i = \int_0^{T_b} f_i(t) f_i(t) \, dt = \delta_{ii},
\]
where \( \delta_{ii} \) is the Kronecker delta. We have the following K-L expansions of the function \( s(t) \) and the processes \( r(t) \), \( w(t) \) and \( w_i(t) \) over \( [0, T_b) \):
\[
\begin{align*}
s(t) &= \sum_{i=1}^{M} s_i(t) f_i(t), \\
r(t) &= \sum_{i=1}^{M} r_i(t) f_i(t), \\
w(t) &= \sum_{i=1}^{M} w_i(t) f_i(t).
\end{align*}
\]
Let \( s_{L, L} = [s_1, \ldots, s_L]^T \), \( r_{L} = [r_1, \ldots, r_L]^T \), \( w_{L} = [w_1, \ldots, w_L]^T \), and \( w_{E} = [w_{E_1}, \ldots, w_{E_L}]^T \). Using (8) and (9), we obtain from (1) the vector model
\[
H : r_{L} = s_{L, L} + E_{L} + W_{L},
\]
where \( s_i \) is a zero-mean real Gaussian random \( i \times 1 \) vector satisfying
\[
E[w_{E} w_{E}^T] = \Sigma,
\]
\( \Sigma \) being the \( L \times L \) identity matrix. Note that \( s_{L, L} \) and \( w_{L} \) are independent random vectors.

The conditional p.d.f. of \( r_{L} \) given \( v \) and hypothesis \( H_0 \) can be expressed using (10) and (11) as
\[
f(r_{L}|v, H_0) = \int f(r_{L}, v, H_0) \, dr_{L}
\]
where \( \langle \cdot \rangle \) denotes the Euclidean norm.

Let the apriori probabilities for the hypotheses be
\[
p_i = P(H_i), \quad i = 0, \ldots, M - 1, \quad \sum_{i=0}^{M-1} p_i = 1.
\]
Using (12), (6) and (4), the likelihood function for hypothesis \( LZ \) is given by
\[
A(r_{L}|H) = \max_{i} \left( \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-1}{2} \left( r_{L} - \mathbf{d}_i \right)^T \Sigma_{i}^{-1} \left( r_{L} - \mathbf{d}_i \right) \right) \right)
\]
where
\[
\mathbf{d}_i = \left\{ D_i, \ldots, D_L \right\}
\]
\( D_i = \frac{1}{T_b} \int_0^{T_b} r(t) s_i(t) \, dt \) (19a)
and
\[
\Sigma_i = \left( \begin{array}{c} \Sigma \end{array} \right) (19b)
\]
where \( \Sigma \) is the noise covariance matrix.

From (17), the decision rule (18) becomes
\[
\text{Choose } i^* = \arg \max_{i} \left( \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-1}{2} \left( r_{L} - \mathbf{d}_i \right)^T \Sigma_{i}^{-1} \left( r_{L} - \mathbf{d}_i \right) \right) \right)
\]
where \( \mathbf{d}_i \) is the \( i \)th decision statistic \( D_i \) given by
\[
D_i = \frac{1}{T_b} \int_0^{T_b} r(t) s_i(t) \, dt + \frac{1}{T_b} \int_0^{T_b} \mathbf{d}_i \, dt \] (19b)
and
\[
\Sigma_i = \left( \begin{array}{c} \Sigma \end{array} \right) (19c)
\]
where \( \Sigma \) is the noise covariance matrix.
The statistic $D$, for the MAP detector is a sum of three terms: a matched filter or correlator term (19ba), a nonlinear term (19bb), and a bias term (19bc). When the arrival rate $X$ of the pulses is zero the nonlinear term vanishes, resulting in the usual MAP decision statistic for coherent M-ary signaling in additive white Gaussian noise.

Denote the kth moment of each of the pulse amplitudes $a_1, a_2, \ldots$, as

$$
\mu^{(k)}_a = \int_{-\infty}^{\infty} x^k f_a(x) dx.
$$

(20)

It is reasonable to assume that each $a_m$ has a symmetric p.d.f., which implies that $f_a(x)$ is an even function, and therefore

$$
\sigma_{W_n}^2 = 0 \quad \text{for } k = 1, 3, 5, \ldots
$$

(21)

Using (21) and the Taylor's series expansion for $\exp(x)$ about the point 0 in (19bb), and substituting

$$
(r(t) - \nu(t))^2 = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!} \nu^n(t) \sigma^n W(t),
$$

(22)

we obtain the following alternative detector structure in terms of powers of the received signal $r(t)$:

Choose $H_i$ if

$$
J = \text{arg} \left\{ \max_{0 \leq j \leq M-1} \left( \int_0^{T_i} r(t) h_{ij}(t) dt \right) \right\}
$$

(23a)

where

$$
h_{ij}(t) = h_{ij}(t) = h_{ij}(t),
$$

(23b)

and the bias term $B$, is given by

$$
B = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sigma_n^2}{\left( \sigma^2 + \sigma^2_W \right)^{n/2}} \nu^{n}(t),
$$

(23c)

IV. COHERENT BINARY SIGNALING

In this case, $M = 2$. Suppose that we perform detection as in (23) without the nonlinearities. The suboptimum linear detector, which can be viewed as a modified matched filter detector, has the structure

$$
H_i \left\{ \int_0^{T_i} r(t) h(t) dt - B \right\} < 0,
$$

(24a)

such that

$$
h(t) = h(t) - h(t), \quad B = B_1 - B_0,
$$

(24b)

where $h(t)$ is given by (23b) and $B$, by (23d). In (24), the received signal is matched to $h(t)$, a nonlinear function of the signaling waveforms $s_1(t)$ and $s_2(t)$, instead of being matched to $s_1(t) - s_2(t)$, which is the usual case. Under hypothesis $H_1$, the matched filter output in (24a) can be expressed using (1) $M$

$$
\int_0^{T_i} r(t) h(t) dt = S_j + W, \quad j = 1, 2
$$

(25a)

where the signal component $S_j$ is given by

$$
S_j = \int_0^{T_i} r(t) h_j(t) dt,
$$

(25b)

and the impulsive noise component $V$ by

$$
V = \sum_{m=1}^{\infty} \sigma_m h_0(t_m),
$$

(25c)

and the white Gaussian noise component $W$ by

$$
W = \int_0^{T_i} W(t) h(t) dt.
$$

(25d)

The component $W$, which is independent of $V$, is a zero-mean real Gaussian random variable with variance

$$
\sigma_W^2 = \sigma_v^2 \int_0^{T_i} h^2(t) dt.
$$

(26)

The probability of error under $H_1$ can be expressed as

$$
P_e(H_1) = \Pr(W < -(s_1^2 + V + B))
$$

$$
= \int_{-\infty}^{\infty} Q \left( \frac{S_1 - B - \frac{z}{2}}{\sigma_W} \right) f_V(v) dv,
$$

(27)

where $f_V(v)$ is the p.d.f. of $V$, and

$$
Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2} dy.
$$

Denoting

$$
S_0 = \int_0^{T_i} s_0(t) h(t) dt,
$$

(28)
the probability of error under \( H_0 \) is given by

\[
P_e(H_0) = \Pr(W > -(S_0 + V - B)) = f_Q(i \frac{B - S_0 - x}{\sigma_w})f(x)dx.
\]  

(29)

The average probability of error is

\[
p_e = P_e(H_0) + P_e(H_1).
\]  

(30)

Since \( f_0(.) \) is an even function, it can be shown that \( f_0(.) \) is even. Therefore the odd moments of \( V \) are zero. The even moments of \( V \) are given by

\[
\mathcal{E}[V^{2n}] = \sum_{n=0}^{\infty} \binom{n}{n} \sum_{m=0}^{n} a_m h(m) \left( \frac{A_{2n}}{2n} \right)^{n-m} \exp\left(-x_T b\right)\exp\left(-x_T b\right)
\]  

(31)

for \( n = 1, 2, 3, \ldots \)

A multinomial expansion of the expectation term in (31) results in

\[
\mathcal{E}[V^{2n}] = \sum_{n=0}^{\infty} \mathcal{E}[V^{2n}] \exp[-X_T T]\left[ \left( A_{2n} \sum_{m=0}^{n} a_m h(m) \right) \right]^{2n}
\]  

(32)

We can write the performance degradation term (33b) as

\[
E_2(\eta) = P_e Q\left( \frac{S_1 - B}{\sigma_w} \right) + P_e Q\left( \frac{B - S_0}{\sigma_w} \right)
\]  

(33a)

where the term (33a) gives the BER without the impulsive component, while (33b) gives the degradation in the BER due to the presence of the impulsive noise. Note that \( \mathcal{E}[V^{2n}] \) \( f_0(.) \) given by (32), and \( Q(\cdot) \), the (2n)th derivative of the Q-function, is given by

\[
Q(\cdot) = \left( \frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\cdot} e^{-\frac{x^2}{2}} dx
\]  

(33b)

Using the symmetry of \( f_0(.) \) and the Taylor's series expansion of \( Q(\cdot) \), we obtain from (30)

\[
P_e = p_0 Q\left( \frac{S_1 - B}{\sigma_w} \right) + p_1 Q\left( \frac{B - S_0}{\sigma_w} \right)
\]

(34a)

The truncation error in (35a), which corresponds to the fourth order term, is given by

\[
E_4(\eta) = \frac{1}{2\sigma_w^2} \mathcal{E}[V^{2n}] \exp[-X_T T]\left[ \left( \frac{A_{2n}}{2n} \right) \right]^{2n}
\]  

(35)

(36)

(37)

Using (31) and the fact that \( \int_{-\infty}^{\cdot} e^{-\frac{x^2}{2}} dx = 0.2419707 \)

\[
\mathcal{E}[Q(\cdot)] \leq \frac{\sqrt{\gamma} - 3 - \sqrt{\gamma}}{\sqrt{2\pi}} = 0.5544948 \]

(38)

and using (31), we obtain an upper bound on the absolute value of the truncation error, which is given by

\[
|E_4(\eta)| \leq \frac{\sqrt{\gamma} - 3 - \sqrt{\gamma}}{\sqrt{2\pi}} \exp[-X_T T]\left[ \left( \frac{A_{2n}}{2n} \right) \right]^{2n}
\]

(39)

Using (31) and the fact that

\[
Q(\cdot) = \left( \frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\cdot} e^{-\frac{x^2}{2}} dx
\]

we can write the performance degradation term (33b) as

\[
E_2(\eta) = \frac{1}{2\sigma_w^2} \mathcal{E}[V^{2n}] \exp[-X_T T]\left[ \left( \frac{A_{2n}}{2n} \right) \right]^{2n}
\]

(40)

Using (31) and the fact that

\[
|Q(\cdot)| \leq \frac{1}{\sqrt{2\pi}} \exp[-X_T T] \left( \frac{A_{2n}}{2n} \right)^{2n}
\]

(41)

(42)
This implies from (23b), (23d) and (24b) that
\[ B = 0, \]
\[ h(t) = 2Is'(t) - dtl, \]
where
\[ g(t) = \lambda \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^k 2^k a^2 (2k+2l)^2}{k(2l-1)^2 n^{2k} (2n+1) - 2^{2k+1}} s^{2k+1}(t). \]
Also
\[ S_1 = 5_0 = 5 = \int_{0}^{T_c} s(t) h(t) dt. \]
From (33), the BER is given by
\[ P_e = Q(\frac{S}{\sigma_w}) + \sum_{n=1}^{\infty} \frac{\epsilon \frac{5^{2n}}{(2n)!} \frac{Q(2n)}{\sigma_w}}{1 + b^2 - 2a}. \]

Now
\[ \frac{|5|}{\sigma_w} = \sqrt{\frac{2}{\sigma_w}} \frac{1 - a}{\sqrt{1 + b^2 - 2a}}, \]
where
\[ a = \int_{0}^{T_c} s(t) g(t) dt, \quad b = \int_{0}^{T_c} s^2(t) dt. \]
By Schwarz's inequality, \( a^2 < b^2 \), and hence
\[ \frac{|5|}{\sigma_w} \bigg|_{h(t) = 2s(t)} > \sqrt{\frac{2}{\sigma_w}} \frac{1 - a}{\sqrt{1 + b^2 - 2a}}. \]
This implies
\[ Q \left( \frac{\sqrt{\frac{2}{\sigma_w}} s(t) dt}{\sigma_w} \right) > Q \left( \frac{\sqrt{\frac{2}{\sigma_w}} s(t) dt}{\sigma_w} \right) \]
When \( pL2 < \epsilon < u \), the variation of \( P_e \) in (45) with \( \Omega \) is controlled by the dominant term \( Q (\&). \) Therefore, when the Gaussian noise dominates over the impulsive noise, we can say from (48) that the performance of a matched filter detector with \( h(t) = 2s(t) \) is superior to that of one with \( h(t) = 2|s(t) - g(t)| \).
We also make the following conjecture for the case of antipodal signaling:
When the impulsive noise dominates over the Gaussian noise, the performance of the modified matched filter detector with \( h(t) = 2s(t) - g(t) \) will be superior to that of the usual matched filter detector with \( h(t) = 2s(t) \).

V. CONCLUSION

An optimum nonlinear detector for coherent M-ary signaling in impulsive noise has been derived using the MAP criterion. For the case of coherent binary signaling, we have analyzed the BER performance using a suboptimum linear modified matched filter detector.

REFERENCES