Analysis of Rake Reception With Multiple Symbol Weight Estimation for Antipodal Signaling

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Abstract—Consider a Rake receiver for coherent binary antipodal signaling with: 1) a delayed received signal configuration; 2) weight estimation by matched filtering using the reference signal along with the decisions of the previous M symbol intervals; and 3) predetection maximal-ratio combining (MRC). The weight estimation errors here are not independent of the additive noise, and do not fit into the Gaussian weighting error model for MRC. Here we analyze the error performance of the receiver by obtaining the conditional symbol error probability, conditioned on past decisions, from the characteristic function of the decision variable, and getting the unconditional error probability (UEP) for a block of M consecutive symbols using a Markov model of the decision process. The channel is Rayleigh fading with independent and identically distributed branch gains. Results show that the error performance of the Gaussian distributed weighting error model is a bound for that of multiple symbol weight estimation by matched filtering, and the steady state UEP decreases with increase of M, but the amount of decrease reduces as M increases.

Index Terms—Binary antipodal signaling, independent and identically distributed (i.i.d.) Rayleigh fading, Markov model, multiple symbol weight estimation, predetection maximal-ratio combining (MRC), Rake reception, symbol error probability (SEP), unconditional error probability (UEP).

I. INTRODUCTION

For wideband signals, such as code-division multiple-access (CDMA) signals, received over a fading channel, a Rake receiver resolves the multipaths via code correlation and then combines them [1]. Predetection maximal-ratio combining (MRC) is the preferred combining technique when the reception is coherent, since it gives the maximum instantaneous signal-to-noise ratio (SNR) at the combiner output. However, MRC requires estimation of the diversity branch gains, scaled versions of which are the tap weights of the Rake receiver. Errors arising from this estimation process degrade the Rake receiver performance. Analysis of MRC with Gaussian distributed estimation errors, which are independent of the additive noise in the channel, has been done in [2]. However, in a Rake receiver for coherent binary antipodal signaling with a delayed received signal configuration, tap weight estimation by matched filtering using the reference signal along with the decisions of the previous M symbol intervals, and predetection MRC (similar to that given in [3, p. 803, Fig. 14-5-5] for M = 1), the estimation errors are not independent of the additive noise, and do not fit into the model of [2]. This calls for a different approach to the analysis of the Rake receiver, and is the focus of this paper. Here we analyze the error performance of the receiver by obtaining the conditional symbol error probability (SEP), conditioned on past decisions, from the characteristic function (cf) of the decision variable, and getting the unconditional error probability (UEP) for a block of M consecutive symbols using a Markov model of the decision process. A Rayleigh fading channel with independent and identically distributed (i.i.d.) branch gains is considered. Results show that the error performance of the Gaussian distributed weighting error model is a bound for that of multiple symbol weight estimation by matched filtering, and the steady state UEP decreases with increase of M; however, the amount of decrease reduces as M increases. The error probability formula derived in this paper can also be used to compute the SEP for binary phase-shift keying (BPSK) systems using multiple receive antennas for diversity reception, where the received signals are subject to i.i.d. Rayleigh fading.

The paper is organized as follows. Analysis of the conditional SEP, conditioned on past decisions, is presented in Section II. Section III deals with the conditional error performance in i.i.d. Rayleigh fading. Using a Markov model of the decision process, the UEP for a block of M consecutive symbols is obtained in Section IV. Section V shows a performance comparison of multiple symbol weight estimation by matched filtering with the Gaussian distributed weight estimation error model of [2]. Numerical results are presented in Section VI. Section VII contains some concluding remarks.

II. SYMBOL ERROR PERFORMANCE

Consider a diversity reception system over a fading channel with L branches, which uses a Rake receiver. The channel is modeled by a tapped delay line with L taps. When symbol $q_i$, $q_i \in \{-1,1\}$, is transmitted in the $i$th symbol interval of duration $T_s$, the complex baseband received signal can be represented as

$$r(t) = \sum_{i=-\infty}^{\infty} q_i s_{qi}(t - iT_s - \frac{t - 1}{W_s}) + n(t), \quad iT_s \leq t < (i+1)T_s$$

(1)

where $s_{qi}(t)$ is the complex baseband information-bearing signal corresponding to symbol $qi$, having support $[0, T_s)$,
average symbol energy $2E_b$ and large bandwidth $W/2$ satisfying $W > L/T_s$. $g_i$ is the random complex gain of the $i$th branch, and $n(t)$, representing the additive noise, is a zero-mean complex circular white Gaussian random process with two-sided power spectral density $2N_0$. The fading is assumed to be frequency selective but time flat. The branch gains $\{g_i\}$ are assumed to be invariant over $M + 1$ consecutive symbol intervals and are independent of the noise $n(t)$.

Let

$$s_2(t) = -s_{t-1}(t) = s(t). \quad (2)$$

The signaling waveform $s(t)$ is a wideband signal generated by a pseudorandom sequence, and satisfies

$$\int_0^{T_s} s\left(t - \frac{k}{W}\right)s^*\left(t - \frac{l}{W}\right)dt = \begin{cases} 2E_b, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases} \quad (3)$$

where $(\cdot)^*$ denotes the complex conjugate. Note that (3) is a simplifying assumption for typical pseudonoise (PN) sequences.

We have made this assumption for analytical simplicity in the derivations that follow.

In the $t$th symbol interval, the Rake receiver computes tap weights $h_{i,i} \ldots h_{L,i}$ that are scaled estimates of the branch gains $g_1 \ldots g_L$, respectively, as in [3, p. 803, Fig. 14-5-5] for $M = 1$, and uses these weights to combine the received signal along with its $L - 1$ delayed versions by MRC. The decision variable $D_i$ resulting from coherent reception and MRC can be expressed as shown in (4) at the bottom of the page. The receiver makes the decision

$$D_i = \begin{cases} +1, & \text{if } h_{i,i} > 0 \\ -1, & \text{otherwise} \end{cases} \quad (4)$$

As shown in Fig. 1, the estimated weights $h_{i,i} \ldots h_{L,i}$ obtained by matched filtering, are given by (5), shown at the bottom of the page, where $s(t - iT_s - (L - i)/W)$ is the reference signal and $\text{sgn}(-)$ denotes the signum or sign function. Note that the integration in (5) corresponds to a low-pass filter.

$$h_{b,i} = \sum_{m=1}^{M} \text{sgn}(D_{i-m}) \int_{iT_s}^{(i+1)T_s} r\left(t - mT_s - \frac{L - k}{W}\right)s^*\left(t - iT_s - \frac{L - 1}{W}\right)dt \quad (5)$$
filtering operation. Thus, in the ith symbol interval, the tap weights are estimated by making use of the decision variables A - i, •••, A - Af of the M previous symbol intervals.

Let the random noise variable Nk be defined as

\[ N_{k_{\text{tot}}} \triangleq \int_{T_k}^{(i+1)T_k} \left[ \sum_{m=1}^{M} g_m^* \right] \mathbf{H}^\dagger \mathbf{A}_{\text{tot}} \mathbf{r}_{\text{tot}} \mathrm{d} t \]

It can be shown that \( \{N_k\} \) are i.i.d. zero-mean complex circular Gaussian random variables with \( \mathbb{E}[|N_k|^2] = 0 \) and variance \( \text{var}[N_k] = 4E_s N_0 \) where \( \text{var}[\cdot] \) denotes the expectation, and \( \mathcal{CN}(0, 4E_s N_0) \) distribution.

Using (5) and (6), the weight estimate \( \mathbf{h}_{k_{\text{tot}}} \) can be expressed in terms of \( N_{k_{\text{tot}}} \), \( i-i_{\text{tot}} \), •••, \( N_{k_{\text{tot}}} \), \( i-M \) as

\[ h_k = \sum_{m=1}^{M} \sum_{l=1}^{\infty} \text{sgn}(A_{\text{tot}}^m) \text{sgn}(A_{\text{tot}}^l) \left[ \text{Res}\left( \frac{1}{\mathbf{Y}_{k_{\text{tot}}}^\dagger \mathbf{Y}_{k_{\text{tot}}} + \frac{1}{2} \mathbf{r}_{\text{tot}}^\dagger \mathbf{r}_{\text{tot}}} \right), j = 1 \right] \]

Defining the random variable \( Y_{kj} \) as

\[ y = \mathbf{g}^\dagger \mathbf{r}_{\text{tot}} \]

and substituting (7) in (8), we can rewrite \( A \) in terms of \( Y_{kj} \), as

\[ D_{k_{\text{tot}}} = \mathbf{Y}_{k_{\text{tot}}}^\dagger \mathbf{A}_{\text{tot}} \mathbf{K} \{ r_{\text{tot}}^\dagger \}
\]

Note that each \( Y_{kj} \), conditioned on \( g_k \), is a complex circular Gaussian random variable having a \( \mathcal{CN}(0, 4E_s N_0) \) distribution. Thus (10) is a Hermitian quadratic form in complex Gaussian random variables.

Let the random complex channel gain vector g be defined as

\[ g = [g_1, \ldots, g_L]^T \]

where \( (\cdot)^T \) denotes transpose. The ideal instantaneous SNR at the combiner’s output, denoted as \( \gamma_{\text{tot}} \), is expressed as

\[ \gamma_{\text{tot}} = \sum_{k=1}^{L} \gamma_k = \frac{E_s}{N_0} \sum_{k=1}^{L} |g_k|^2 = \frac{E_s}{N_0} \mathbf{g}^\dagger \mathbf{g} \]

where \( (\cdot)^H \) denotes the Hermitian (conjugate transpose) operator and \( \gamma_k = (E_s/N_0) g_k^2 \) is the ideal instantaneous SNR at branch k. Using properties of Hermitian quadratic forms in complex Gaussian random variables \([4]\), we have shown in Appendix A that the cf of A, conditioned on \( j_{i_{\text{tot}}} \), is given by

\[ \mathbf{S}_{\mathbf{A}_{j_{i_{\text{tot}}}}}(\mathbf{r}_{\text{tot}}^\dagger) \]

\[ = \mathbf{E}[\exp(jwA^\dagger \mathbf{r}_{\text{tot}}^\dagger)] \]

\[ \frac{4\pi^2 E_s N_0 f_s + \pi^2 E_s N_0 (M + f_s^2)}{1 + 4Mw^2 \mathbf{E}^2_s \mathbf{N}_0^2} \]

(13a)

where

\[ j = \sum_{m=1}^{M} g_m^* \text{sgn}(A_{\text{tot}}^m) \in \{-1, 1\} \]

and \( j \) is the number of times \( \text{sgn}(A_{\text{tot}}^m) \) takes the value one, or the number of correct decisions between the \( (i - M) \)th symbol intervals.

Let the cf of \( \gamma_{\text{tot}} \) be denoted as \( *_{\text{tot}}(j\omega) \). Averaging \( \mathbf{E}[\exp(jwA^\dagger \mathbf{r}_{\text{tot}}^\dagger)] \) using the inversion theorem \([5]\),

\[ \mathbf{E}[\exp(jwA^\dagger \mathbf{r}_{\text{tot}}^\dagger)] \]

\[ \frac{4\pi^2 E_s N_0 f_s + \pi^2 E_s N_0 (M + f_s^2)}{1 + 4Mw^2 \mathbf{E}^2_s \mathbf{N}_0^2} \]

(13a)

The conditional SEP at the ith symbol interval \( \gamma_{i_{\text{tot}}} \) conditioned on \( q_i \) and \( f_i \), which we denote as \( \text{Pe}(q_i, f_i) \), is given by

\[ \text{Pe}(q_i, f_i) = \frac{1}{\gamma_{i_{\text{tot}}}} \mathbf{E}[\exp(jwA^\dagger \mathbf{r}_{\text{tot}}^\dagger)] \]

\[ \frac{4\pi^2 E_s N_0 f_s + \pi^2 E_s N_0 (M + f_s^2)}{1 + 4Mw^2 \mathbf{E}^2_s \mathbf{N}_0^2} \]

(15)

Formulae for \( \text{Pe}(q_i, f_i) \), \( q_i = 1, -1 \), can be obtained directly from the cf \( *_{\text{tot}}(j\omega) \) using the inversion theorem \([5]\). After changing the variable \( \omega \) to \( z = \frac{2\pi}{\omega} \), in (15), we get

\[ \text{Pe}(q_i, f_i) = \frac{1}{\gamma_{i_{\text{tot}}}} \mathbf{E}[\exp(jwA^\dagger \mathbf{r}_{\text{tot}}^\dagger)] \]

\[ \frac{4\pi^2 E_s N_0 f_s + \pi^2 E_s N_0 (M + f_s^2)}{1 + 4Mw^2 \mathbf{E}^2_s \mathbf{N}_0^2} \]

(16)

at poles on left-half \( z \)-plane
where, from (15), we have
\[
\Psi_V\left(\frac{z}{2E_N},1,f_i\right) = \Psi_V\left(\frac{z}{2E_N}, i, f_i\right) = \frac{z}{z - a_i} \left(\frac{a_i + b_i}{M + MT + ffr}\right)^L.
\] (21)

Substituting (21) into (15), we obtain
\[
\Psi_D\left(\frac{z}{2E_N},1,f_i\right) = \frac{(-1)^L}{z(z + a_i)^L(z - b_i)^L(M + MT + ffr)^L}.
\] (22)

where
\[
a_i = \frac{\sqrt{(1 + r)}(M + ffr) + ffr}{M + M + \Gamma + ffr},
\] (23)

Note that \(a^*\) and \(b^*\) are both positive quantities. It is clear from (22) that \(V_{Di}(z/2E_N,0,1, f_i)\) has only one pole of order \(L\) at \(z = -a_i\) on the left-half \(z\)-plane, while \(f_{fi}(z/2E_N,0,1, f_i)/z\) has only one pole of order \(L\) at \(z = a^*\) on the right-half \(z\)-plane. Therefore, substitution of (22) in (17) yields
\[
P_{ei}(1, f_i) = -\frac{1}{(L-1)!} \times \frac{d^{L-1}}{dz^{L-1}} \left[ \frac{(-1)^L}{z(z + h)^L(M + MT + ffr)^L} \right]_{z = -a_i},
\] (24a)

By changing the variable of differentiation in (24b) from \(z\) to \(z' = -z\), we find that \(P_{ei}^<(-1, f_i) = P_{ei}^>(-1, f_i)\). Hence, the conditional SEP at the \(j\)-th symbol interval, conditioned on \(f_i\), which we denote as \(P_{ci}(f_i)\), is given by
\[
P_{ci}(f_i) = P_{ei}(1, f_i) - P_{ei}(-1, f_i).
\] (26)

By using Faas di Bruno’s formula [6] to evaluate the \((L-1)\)th derivative as in [7], we obtain from (24) and (26) the expression
\[
P_{ei}(f_i) = \frac{1}{(M + MT + ffr)^L} \times \frac{1}{n (a^* + b^*)} \left(\frac{L}{n - 1}\right),
\] (27)

where the summation is over all \((L-1)\)-tuples \((i_1, \ldots, i_{L-1})\) of integers in the range \([0, L-1]\) satisfying \(J2^{i_1} - J_i \text{mod} M = L-1\), and \(a_i\) and \(b_i\) are given by (23). Thus, (27) is a closed-form expression for the conditional SEP in i.i.d. Rayleigh fading.

Note that the conditional SEP \(P_{ei}(f_i)\) depends on the value of \(f_i\), which ranges from \(-M\) to \(M\) at intervals of two \(\text{[given by (14)]}\), but has no direct dependence on \(i\). We therefore define a probability \(p_j\) as
\[
p_j = P_{ei}(2j - M), \quad j = 0, \ldots, M.
\] (28)

For any symbol interval, the subscript \(j\) of \(p_j\) is the number of correct decisions made over the past \(M\) symbol intervals. Thus, \(p_j\) is the conditional SEP in any symbol interval, conditioned on the fact that \(j\) correct decisions have occurred over the past \(M\) symbol intervals. Substituting (23) in (27), we can rewrite \(p_j\) as
\[
p_j = \frac{(M + MT + (2j - M)^2T)^L}{\left(2\sqrt{(1 + \Gamma)}(M + (2j - M)^2T)\right)^L} \times \frac{1}{\left(\sqrt[1+r]{(1+r)(M+(2j-M)^2T)} + (2j-M)T\right)} \times \sum_{\left(i_1, \ldots, i_{L-1}\right) \in \mathbb{Z}} \frac{1}{2^L \prod_{j=1}^{L-1} i_j \prod_{j=1}^{L-1} \left(a^* + b^*\right)},
\] (29)
We have from (26) and (28) the relation

\[ \frac{M}{2} \cdot \frac{1}{2} = \frac{1}{2} \quad \text{when } M \text{ is even.} \]

IV UNCONDITIONAL ERROR PROBABILITY (UEP)

Consider the \( M \times 1 \) decision vector \( D_q^i \) defined as

\[
D_q^i = \begin{bmatrix}
\sum_{m=1}^{M} q_i - m + 1 \cdot \text{sgn}(D_{i-1,m+1}) \\
\sum_{m=1}^{M} q_i - m \cdot \text{sgn}(D_{i-1,m}) \\
\vdots \\
\sum_{m=1}^{M} q_i - m + 1 \cdot \text{sgn}(D_{i-1,M+1}) \\
\sum_{m=1}^{M} q_i - m \cdot \text{sgn}(D_{i-1,1})
\end{bmatrix}
\]

which constitutes the decision of the \( i \)th symbol interval \( iT_i \leq x < (i+1)T_i \) and the decisions of the past \( M - 1 \) symbol intervals. The entries of \( D_q^i \) are either zeros or ones, zero implying an erroneous decision, and one a correct decision. We treat the vector \( D_q^i \) as a binary number, the first row entry representing the most significant bit, and the \( M \)th row entry the least significant bit. The decimal equivalent of \( D_q^i \), which we denote as \( C^i \), can be expressed as

\[ \sum_{m=1}^{M} c_{i,m} = 2^m \cdot \text{decimal equivalent} \]

Thus, \( c_{i,m} \) ranges from 0 to \( 2^{M-1} \), and each value of \( c_i \) maps onto a corresponding value of \( f_{i+1} \), where from (13b)

\[ f_{i+1} = \sum_{m=1}^{M} q_i - m + 1 \cdot \text{sgn}(D_{i-1,m+1}). \]

Note that \( f_{i+1} \) represents the number of ones in \( D_q^i \) minus the number of zeros in \( D_q^i \).

and that different values of \( C_i \) can map onto the same value of \( f_{i+1} \). However, for every value of \( Q_i \), there is a unique value of \( f_{i+1} \). For example, when \( M = 2 \), we have \( c_i \in \{ 0, 1, 2, 3 \} \), while \( f_{i+1} \in \{ -2, 0, 2 \} \). In this case, \( c_i = 0 \) maps onto \( f_{i+1} = 0 \), \( c_i = 1 \) maps onto \( f_{i+1} = 2 \), \( c_i = 2 \) maps onto \( f_{i+1} = 0 \), and \( c_i = 3 \) maps onto \( f_{i+1} = 2 \).

A reasonable measure of the UEP is the probability that at least one error occurs in a block of \( M \) consecutive symbols. The UEP at the \( i \)th symbol interval can be expressed as

\[ P_{uei} = 1 - \Pr (f_{i+1} = M). \]

We proceed to find a way of obtaining the UEP.

If in the \( (i - 1) \)th symbol interval, \( Q_{i-1} = c_1 \), then in the \( i \)th symbol interval, we have \( c_i = \lceil c/2 \rceil \) in case of an error, and \( c_i = 2^{M-1} + \lceil c/2 \rceil \) in case of a correct decision. Let \( C_{i-1} = c \) map onto \( f_{i+1} = 2^j - M \). We can then express the conditional probability of \( c_i \) given \( C_{i-1} = c \) as

\[ \Pr (c_i = \lceil c/2 \rceil | c_{i-1} = c) = \Pr (q_i \text{sgn}(D_i) < 0 | f_i = 2^j - M) = P_e(c_i = 2^j - M) = p_j \]

\[ \Pr (c_i = 2^{M-1} + \lceil c/2 \rceil | c_{i-1} = c) = \Pr (q_i \text{sgn}(D_i) > 0 | f_i = 2^j - M) = 1 - P_e(c_i = 2^j - M) = 1 - p_j \]

where \( p_j \) is given by (29). It is clear that the random variable \( C_i \), which takes integer values from 0 to \( 2^M - 1 \), depends only on the immediate past variable \( c_{i-1} \), and not on the earlier variables. In addition, the conditional probability of \( C_i \) given \( c_{i-1} \) does not depend on \( i \). As a result, the random sequence \( \{ c_i \} \) is a discrete-time discrete-state homogeneous Markov process, having the set \( \{ 0, 1, 2, \ldots, 2^M - 1 \} \) of \( 2^M \) states.

Let \( P_{s}(\cdot) \) denote the state probability vector in the \( i \)th symbol interval, given by

\[ P_{s}(i) = \begin{bmatrix}
P_{s_0}(i) \\
P_{s_1}(i) \\
\vdots \\
P_{s_{2^M-1}}(i)
\end{bmatrix} = \begin{bmatrix}
\Pr (c_i = 0) \\
\Pr (c_i = 1) \\
\vdots \\
\Pr (c_i = 2^M - 1)
\end{bmatrix} \]

The \( 2^M \times 2^M \) transition probability matrix, denoted as \( P_T \), with rows indexed by states of \( c_i \) and columns by states of \( c_{i-1} \) in the order \( 0, 1, 2, \ldots, 2^M - 1 \), can be easily constructed using (36). Thus, we have the relation

\[ P_s(0) = P_T P_s(i - 1) = P_T P_s(0) \]

where the initial state probability vector \( P_s(0) \) is a known quantity.

We begin operating the receiver by receiving a known training sequence \( q_0, q_1, \ldots, q_{M-1} \). This implies \( q_i = \text{sgn}(A) \) for \( i = 0, 1, 2, \ldots, M - 1 \). Thus, \( \Pr (c_0 = 2^M - 1) = 1 \), implying

\[ P_s(0) = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix} \]

From (38) and (39), we can get \( P_{s}(\cdot) \) in terms of the entries of \( P_T \). The UEP is given by

\[ P_{uei} = 1 - P_{s_2^{M-1}}(i) \]

which is in terms of \( P(0 M/2j) \).

In the case of \( M = 1 \), the states are 0, 1

\[ \begin{bmatrix}
0 \\
1 - P_0 \\
P_0
\end{bmatrix} \]

and it can be shown that

\[ P_{uei} = 1 - P_{s_2(i)}(i) = P_{s_2(i)}(i) = 1 - \frac{2^{2n} - 1}{2}. \]
In the case of $M = 2$, the states are 0, 1, 2, 3

$$PT = \begin{bmatrix} P_0 & P_l & 1-P_0 \\ 0 & 0 & 1-P_l \\ 0 & 0 & P_0 \end{bmatrix}$$

(42a)

with $p_l = 1/2$, and we can show that

$$PU^* = 1 - P_{S3(i)}$$

(42b)

where

$$\beta_1 = \frac{(2p_0 - 1) + \sqrt{(2p_0 - 1)(2p_0 + 7)}}{4}$$

(42c)

$$\beta_2 = \frac{(2p_0 - 1) - \sqrt{(2p_0 - 1)(2p_0 + 7)}}{4}$$

$$f_{h2} = \frac{(2p_0 - 1) + \sqrt{(2p_0 - 1)(2p_0 + 7)}}{4}$$

$$B_1 = \frac{(\beta_1 - 1)(\beta_1 - 2p_0 - 1)(2p_0 - 1) - \beta_1}{4p_0 - 2}$$

$$D_2 = \frac{(\beta_2 - 1)(\beta_2 - 2p_0 - 1)(2p_0 - 1) - \beta_2}{4p_0 - 2}$$

V. COMPARISON WITH EARLIER MODEL

In [2], the errors in the tap weight estimates $\hat{h}_i, \ldots, \hat{h}_H$ are modeled as complex Gaussian random variables. These errors are characterized by the squared correlation between $h_k$ and $g_k$. In our framework, this squared correlation can be denoted as $p_{h}^{2}$ and is given by

$$p_{h}^{2} = \frac{\left|2ME_{M} E|h_{i1}^{2}g_{i1}^{2}|\right|^2}{E[h_{i1}^{2}g_{i1}^{2}]}$$

(43)

It can be shown from (7), (13b), and the statistics of $N_{k,i}$ and $9k$ that

$$E[h_{i1}g_{i1}^{2}] = 2E_{M} \Omega_{i1}^{2},$$

$$E[f_{i1}^{2}] = 4E_{M} N_{0} + 4E_{M} \Omega_{i1}^{2}.$$  

(44)

As a result, we get

$$E[h_{i1}g_{i1}^{2}] = 2E_{M} \Omega_{i1}^{2}(2j - M) \Pr(f_i = 2j - M)$$

(46)

$$E[f_{i1}^{2}] = 4E_{M} N_{0} + 4E_{M} \Omega_{i1}^{2}$$

(47)

We compare the SEP $P_{e,\text{Gaus}}$ in (47) with the unconditional SEP (USEP) in the $i$th symbol interval, which, from the structure of the state probability vector in (37), can be expressed as

$$P_{e,\text{Gaus}} = \sum_{k=1}^{L} A(k,i) \int \frac{1}{\pi} \left( \frac{\sin^{2} \theta}{\Gamma + \sin^{2} \theta} \right)^{k} \sin \theta d\theta$$

(48)

where the weighting coefficient $A(k,i)$ can be expressed in terms of $p_{h}^{2}$ as

$$A(k,i) = \left( \frac{L-l}{L-1} \right)^{1-k} \left( 1 - p_{h}^{2} L \beta_{1} \right)^{2(k-1)}.$$  

(49)

The integral $\int \frac{1}{\pi} \left( \frac{\sin^{2} \theta}{\Gamma + \sin^{2} \theta} \right)^{k} \sin \theta d\theta$ in (47a) can be written in closed form as

$$\frac{1}{\pi} \int \frac{1}{\pi} \left( \frac{\sin^{2} \theta}{\Gamma + \sin^{2} \theta} \right)^{k} d\theta$$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\Gamma}{1+\Gamma}} \sum_{l=0}^{k-1} \left( \frac{2l}{l} \right) \frac{1}{4l(1+\Gamma)^{l}}.$$  

(50)

We compare the SEP $P_{e,\text{Gaus}}$ in (47) with the unconditional SEP (USEP) in the $i$th symbol interval, which, from the structure of the state probability vector in (37), can be expressed as

$$P_{e,i} = \sum_{i=0}^{M-1} P_{R_{i}}.$$  

(51)

It is clear from (40) and (49) that $P_{i} \leq P_{\text{ave}}$ and that $P_{i} = P_{\text{ave}}$.  

In Figs. 2 and 3, plots of the USEP $P_{e,i}$ versus $V$ resulting from weight estimation by matched filtering are compared with those of the SEP $P_{e,\text{Gaus}}$ versus $T$ resulting from the Gaussian distributed weighting error model of [2]. The fading is assumed to be i.i.d. Rayleigh. We have computed $P_{e,i}$ using (49) and $P_{e,\text{Gaus}}$.
Fig. 3. Performance comparison of weight estimation by matched filtering with Gaussian distributed weighting error model for $M = 2, L = 8$.

using (47). The value of $i$, indicates the number of symbols that are received between two successive training phases. The USEP $P_i$ increases with increase of $i$, while the SEP $P_{e_i, \text{Gaus}}$ increases with increase of $i$ for small values of $F$ but does not change with $i$ for large values of $F$. We also find that

$$-P_{50} > P_{e_i, \text{Gaus}} > P_{e_i, \text{Gaus}^+} > P_i.$$ 

Therefore, $P_{e_i, \text{Gaus}}$ gives an upper bound of $P_i$ for small values of $i$ and a lower bound of $P_i$ for large values of $i$. We also observe that an increase in the estimation block size $M$ from one to two causes the performance to improve considerably. For example, at $T = 10$ dB, the USEP $P_i$ for $M = 1, i = 50$ in Fig. 2 is $10^{-5.4}$, while that for $M = 2, i = 50$ in Fig. 3 is $10^{-4.4}$, implying an improvement by a factor of 10. Similarly, the SEP $P_{e_i, \text{Gaus}}$ at $F = 10$ dB for $M = 1, i = 50$ is $10^{-6}$, while that for $M = 2, i = 50$ is $10^{-5.2}$.

To demonstrate the accuracy of our analytical results, we compare UEP and USEP values obtained by computation using (40) and (49) with those obtained by simulation in Table I. We consider i.i.d. Rayleigh fading with $F = 2$ dB, $M = 1, 2, L = 8$. Each simulation result has been obtained from 73 000 runs, and, for each run, the channel gain $g_k$ is assumed invariant over the entire block of data symbols between two successive training phases. The relative error for each UEP or USEP is given by the equation at the bottom of the page. We see from Table I that the relative error is within 3.8%, implying that the simulation results are very close to the computed results.

VI. NUMERICAL RESULTS

Plots of the UEP in i.i.d. Rayleigh fading, computed using (40), are shown in Figs. 4, 5, and 6. The variation of the UEP

### Table I

<table>
<thead>
<tr>
<th>$M = 1, i = 8, r = 2$ dB</th>
<th>Computation</th>
<th>Simulation</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>UEP $p_{e_i}$ ($= \text{UEP}<em>{p</em>{e_i}}$)</td>
<td>$3.4071 \times 10^{-3}$</td>
<td>$3.5343 \times 10^{-3}$</td>
<td>3.73%</td>
</tr>
<tr>
<td>USEP $p_{e_i}$ ($= \text{USEP}<em>{p</em>{e_i}}$)</td>
<td>$1.4478 \times 10^{-1}$</td>
<td>$1.3973 \times 10^{-1}$</td>
<td>3.49%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$M = 0, r = 2$ dB</th>
<th>Computation</th>
<th>Simulation</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>UEP $p_{e_i}$ ($= \text{UEP}<em>{p</em>{e_i}}$)</td>
<td>$8.9752 \times 10^{-4}$</td>
<td>$8.7671 \times 10^{-4}$</td>
<td>2.32%</td>
</tr>
<tr>
<td>USEP $p_{e_i}$ ($= \text{USEP}<em>{p</em>{e_i}}$)</td>
<td>$2.9250 \times 10^{-2}$</td>
<td>$2.8301 \times 10^{-2}$</td>
<td>3.24%</td>
</tr>
</tbody>
</table>
with the symbol interval index $i$ for $M = 1, \ldots, 8$, $L = 8$, and $F = 6$ dB is shown in Fig. 4. The UEP increases with increase in $i$ and tends to saturate to a steady state value. We find that for $M = 1$, we can receive only up to 25 symbols between two successive training phases to keep the UEP within a reasonable limit of $1(T^3)$. For $M = 2$, the UEP is within $1(T^4)$ when $i < 25$, implying better performance. For $M = 3, \ldots, 8$, the UEP saturates to steady state values less than $10^{-5}$ at $i < 8$, from which we can conclude that use of training symbols at frequent intervals is not needed when $M > 3$. Although the steady state decreases with increase of $M$, the amount of decrease reduces as $M$ increases. Therefore, though there is a distinct advantage in increasing $M$ beyond 2, there is not much advantage in going for very high values of $M$.

Fig. 5 shows the plot of the UEP versus $i$ for $M = 4, T = 6$ dB, and $L = 6, \ldots, 10$. We find that the steady state value of the UEP decreases by about $10^{-5}$ when $L$, the number of branches or Rake fingers, increases by one.

A plot of the steady state UEP versus $T$ for $L = 8, 9, 10$, and $M = 3, \ldots, 6$ is shown in Fig. 6. The steady state UEP decreases with increase of $T$, $L$, or $M$; however, the amount of decrease in the steady state UEP with increase of $M$ diminishes as $M$ goes from three to six. There is not much advantage in using $M = 6$ as compared to using $M = 4$. So, when $L \geq 8$, we need not go beyond $M = 4$.

VII. CONCLUSION

We have analyzed the error performance of a Rake receiver for antipodal signaling in which the tap weights are estimated by matched filtering using the reference signal along with the fading. Next, a Markov model of the decision process, with transition probabilities expressed in terms of the conditional SEP, is formulated. The UEP for a block of $M$ consecutive symbols is obtained from the Markov model. It is found that for small values of $i$, the number of symbols received between two successive training phases, the error performance of the Gaussian distributed weighting error model is an upper bound for that of multiple symbol weight estimation by matched filtering, while for large values of $i$, it is a lower bound; and the steady state UEP decreases with increase of $M$, but the amount of decrease reduces as $M$ increases.

APPENDIX

Define the $(M + 1) \times 1$ random vector $Y_{k;i}$ as

$$Y_{k;i} \triangleq [Y_{k,i}, Y_{k,i-1}, \ldots, Y_{k,i-M}]^T$$

(50)

where $Y_{k,i}$ is defined in (9), and the symbol vector $q_i$ as

$$q_i \triangleq [q_i, q_{i-1}, \ldots, q_{i-M}]^T.$$ 

(51)

The vector $Y_{f;i}$, conditioned on the $f$th branch gain $g_{f;i}$, is a complex circular Gaussian random vector satisfying

$$E[|Y_{k;i}|^2] = 2E_q g_{f;i} q_i,$$

$$E[(Y_{f;i} - 2E_q g_{f;i} q_i)(Y_{f;i} - 2E_q g_{f;i} q_i)^H] = 0,$$

$$E[(Y_{f;i} - 2E_q g_{f;i}) (Y_{k;i} - 2E_q g_{f;i})^H] = 4E_q N_d m_{f;i}$$

(52)

where $Ly/4i$ denotes the $(M + 1) \times (M + 1)$ identity matrix. Thus, $Y_{f;i}$ has a $CAf(2E_q g_{f;i} q_i, AE_q N_d m_{f;i})$ distribution. In addition, we define the decision variable matrix $D_i$ as

$$D_i \triangleq \begin{bmatrix} o & \text{sgn}(A-i) & \cdots & \text{sgn}(A-M) \\ \text{sgn}(A-i) & o & \cdots & o \\ \text{sgn}(A-2) & o & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \text{Lsgn}(A-M) & 0 & \cdots & 0 \end{bmatrix}$$

(53)

Substituting (50) and (53) in (10), we get

$$D_i = \frac{1}{2} \sum_{f=1}^{F} Y_{f;i}^H D_i Y_{k;i}.$$ 

(54)

Owing to the independence of the random noise variables $N_{f;i}, \ldots, N_{f;i}$ given by (6), the vectors $Y_{k;i}, \ldots, Y_{k;i}$, conditioned on $g_{f;i}$, are independent.

Using properties of Hermitian quadratic forms in complex Gaussian random variables [4], the cf of $D_i$, conditioned on the ideal instantaneous combiner output SNR $\gamma_{tot} = (E/N_0) E[1]$ for $i = 0, \ldots, 5$, given by

$$\Psi_{D_i} = \exp\left\{j\omega E_s N_0 \gamma_{tot} q_i^H D_i (1_{M+1} - 2j\omega E_s N_0 D_i)^{-1} q_i \right\} \det (1_{M+1} - 2j\omega E_s N_0 D_i)$$

(55)

We can express the matrix Im$+\mathbf{I} - 2j\omega E_s N_0 D_i$ in (55) as

$$I_{M+1} - 2j\omega E_s N_0 D_i = \begin{bmatrix} 1 & d^T \\ \bar{d}; & \mathbf{1}_M \end{bmatrix}$$

(56a)
where the $M \times 1$ vector $\mathbf{d}_i$ is defined by

$$
\mathbf{d}_i = \begin{bmatrix}
\text{sgn}(A-i) \\
\text{sgn}(A-2) \\
\vdots \\
\text{sgn}(A-M)
\end{bmatrix},
$$

(56b)

It can be shown that

$$
\det \left( \begin{bmatrix}
1 & \mathbf{d}_i^T \\
\mathbf{d}_i^T & I_M
\end{bmatrix} \right) = 1 - \mathbf{d}_i^T \mathbf{d}_i
$$

(57a)

and

$$
lit \mathbf{I}^{-1} = \frac{1}{1 - \text{df} \mathbf{d}_i} \begin{bmatrix}
1 & -\mathbf{d}_i^T \\
\mathbf{d}_i & 1
\end{bmatrix} I_M + \mathbf{d}_i \mathbf{d}_i^T
$$

(57b)

Substituting (57b) in (55) and simplifying the resulting expression, we obtain (13).

REFERENCES


