Analysis of H-S/MRC in Correlated Nakagami Fading

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Abstract — We present an analysis of a hybrid selection/maximal-ratio combining (H-S/MRC) diversity system over an evenly correlated slow frequency-nonselective Nakagami fading channel. In this system, the L branches with the largest instantaneous signal-to-noise ratio (SNR) out of N available branches are selected and combined using maximal-ratio combining. From the joint characteristic function (c.f.) of the instantaneous branch SNR's, we obtain an expression for the c.f. of the combiner output SNR as a series of elementary c.f.'s. The expression can be conveniently used to obtain the symbol error probability of coherent detection of different M-ary modulation schemes.

I. INTRODUCTION

Diversity combining is a method by which multiple replicas of the same information-bearing signal received over different diversity branches are combined in some manner in order to combat the adverse effects of fading in wireless systems [1]. Though a high diversity order is possible in many situations, it may not be feasible to use all of the available diversity branches [2, 3]. This has motivated the study of reduced-complexity diversity combining techniques that select the L best branches (from N available diversity branches) and then combine the selected subset of branches based on a chosen criterion. Here, we consider a hybrid selection/maximal-ratio combining (H-S/MRC) diversity system which selects the L branches with largest instantaneous signal-to-noise ratio (SNR), and then combines these branches using maximal-ratio combining (MRC).

H-S/MRC is considered an efficient means to combat multipath fading [4, 5, 6, 7, 8]. This technique has been analyzed when the diversity branch gains are independent, considering both identically distributed and non-identically distributed cases, under Rayleigh and Nakagami fading conditions [4, 5, 6, 7, 8]. However, the situation of non-independent diversity branches, which a - , for example, in spatial diversity due to the correlations between the signals received by the diversity antennas, has been analyzed previously for MRC [9] and equal-gain combining (EGC, both postdetection EGC and predetection EGC) [10, 11] systems but not for H-S/MRC systems.

In this paper, we analyze the case of H-S/MRC in an evenly correlated slow frequency-nonselective Nakagami fading channel, where (1) the correlation coefficient between any pair of the branch gain amplitudes is the same, and (2) all average branch SNR's are equal. Such a correlation model is appropriate when, for example, we use space diversity with closely packed diversity antennas [9]. Symmetrically placed diversity antennas, like a 3-antenna system with each antenna at the vertex of an equilateral triangle, or a Cantenna system with each antenna at the vertex of a regular tetrahedron can fit into this model.

From the joint characteristic function (c.f.) of the instantaneous branch SNR's, we obtain an expression for the c.f. of the combiner output SNR as a series of elementary c.f.'s. The expression can be conveniently used to obtain the symbol error probability (SEP) of coherent detection of several M-ary modulation schemes. We illustrate our methodology using M-my phase-shift keying (MPSK) as an example.

II. SYSTEM MODEL

Consider a diversity reception system for a digitally modulated signal over a Nakagami fading channel with N correlated branches, in which the complex baseband signal received over the kth diversity branch in a symbol duration $T_s$ is given by

$$s(t) = a_k e^{-j ks(t)} + n(t), \quad k = 1, \ldots, N, \quad 0 \leq t < T_s.$$  \hfill (1)

where $s(t)$ is the complex baseband information-bearing signal with average symbol energy $2E_s$. $a_k$ is the random magnitude and $s(t)$ the random phase of the kth diversity branch gain, and $n(t)$, representing the additive noise, is a zero-mean complex white Gaussian random process with a sided power spectral density $2N_0$. The noise processes $n(1), \ldots, n(N)$ are independent but the channel gain magnitudes $a_1, \ldots, a_N$ are correlated Nakagami random variables each with fading parameter $m$. Furthermore, each $n(t)$ is independent of the complex gains $a_1 e^{-j \theta_1}, \ldots, a_N e^{-j \theta_N}$ for $k = 1, \ldots, N$.

>By "slow" we mean that the channel is time-invariant with a symbol duration.
Let $r_k$ denote the instantaneous SNR of the $k$th diversity branch, defined as

$$ r_k^* \triangleq \frac{E}{N_0} r_k, \quad k = 1, \ldots, N. \quad (2) $$

The marginal density of $y_k$, which is a gamma probability density function (p.d.f.), is given by

$$ f_{y_k}(x) = \frac{1}{E r_k} \left( \frac{m}{r_k} \right)^{m/n} e^{-x/m} \gamma^m_{E}, \quad x > 0, \quad (3) $$

where the fading parameter $m$ ($m \geq n$) denotes the Nakagami family. $\gamma(.)$ denotes the gamma function defined by $\gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$, $S_\alpha = \int_0^\infty e^{-t} t^\alpha dt$, $E(.)$ being the expectation operator, and

$$ r_k^* = \frac{E}{N_0} r_k. \quad (4) $$

is the average received SNR of the $k$th diversity branch.

As in [12], we assume that $m$ is a positive integer.

We consider the case when $r_k = r$ for $k = 1, \ldots, N,$ and each pair $(r_k, a)$ has the same coefficient of correlation $r_{k,a}$ that is,

$$ E \{ y_k - E \{ y_k \} \} = r_{k,a} \quad (5) $$

This implies $7_1, \ldots, 7_N$ are exchangeable random variables, meaning that the joint p.d.f. $f_{y_{1}, \ldots, y_{N}}(y_1, \ldots, y_N)$ is the same for all $N!$ permutations $(i_1, \ldots, i_N)$ of $(1, \ldots, N)$. The joint c.f. of the instantaneous SNR vector $7$ defined as $7 \triangleq \{ r_{1}, \ldots, r_{N} \}$, $(.)^T$ denoting transpose, $\psi_{\alpha}(\beta)$ is given by

$$ \psi_{\alpha}(\beta) = \left( \frac{1}{S_{m/2}} \right)^{m/2} \left( \frac{1}{\beta} \right)^{m + 1 - \frac{1}{2} \alpha} \Gamma\left( \frac{m + 1 - \frac{1}{2} \alpha}{2} \right) \times \Gamma\left( \frac{m - \frac{1}{2} \alpha}{2} \right) \times \Gamma\left( \frac{m + \frac{1}{2} \alpha}{2} \right), \quad (6) $$

where $\Gamma\left( \frac{m + \frac{1}{2} \alpha}{2} \right)$ is the univariate gamma p.d.f. with parameter $m + \frac{1}{2} \alpha$ and mean $m + \frac{1}{2} \alpha$. Using the result $[13]$

$$ \psi_{\alpha}(\beta) = \frac{1}{\Gamma(\alpha)} \left\{ \int_0^\infty u^{\alpha-1} e^{-u} du \right\} \left\{ \int_0^\infty \frac{\exp(-u)}{u^{\alpha/2}} du \right\}, \quad (7) $$

we can conclude that each term $\psi_{\alpha}(\beta)$ is the c.f. of an elementary univariate gamma p.d.f. $9(uk; k + m, \frac{1}{2} \alpha)$.

Therefore (12) gives an expression for the joint c.f. of exchangeable gamma random variables as a series of products of elementary univariate gamma c.f.'s.

A combination of (12) and (14) yields an expression for the multivariate p.d.f. of exchangeable gamma random variables, which is

$$ f_{\alpha}(u) = f_{\alpha_1, \ldots, \alpha_N}(u_1, \ldots, u_N), \quad (8) $$

It can be shown that the inverse of $K$ is given by

$$ (K^{-1})_{kl} = \begin{cases} A & \text{if } k = l, \\ B & \text{if } k \neq l, \quad k, l = 1, \ldots, N, \quad (10a) \end{cases} $$

where

$$ A = \frac{m^{(n-1)k}}{m^{(n-1)k} - m}, \quad B = - \frac{m^{(n-1)k}}{m^{(n-1)k} + m}. \quad (10b) $$

Substituting (10) and (11) in (9), and applying the formal power series expansion of $(1 + \%)^{-1}$ along with the multinomial expansion of $E_{i=1}^N \psi_i$, we finally obtain from (9) the expression

$$ \Psi\alpha(\beta) = \left( \frac{1}{S_{m/2}} \right)^{m/2} \left( \frac{1}{\beta} \right)^{m + 1 - \frac{1}{2} \alpha} \Gamma\left( \frac{m + 1 - \frac{1}{2} \alpha}{2} \right) \times \Gamma\left( \frac{m - \frac{1}{2} \alpha}{2} \right) \times \Gamma\left( \frac{m + \frac{1}{2} \alpha}{2} \right). \quad (9) $$

Let

$$ g(u; k, \alpha) \triangleq \left\{ \begin{array}{ll} \frac{1}{u} & \text{if } u > 0, \\ 0 & \text{if } u = 0, a > 0, k = 1, 2, \ldots, (13) \end{array} \right. $$

denote a univariate gamma p.d.f. with parameter $k$ and mean $ka$. Using the result $[15]$

$$ \psi_{\alpha}(\beta) = \frac{1}{\Gamma(\alpha)} \left\{ \int_0^\infty u^{\alpha-1} e^{-u} du \right\} \left\{ \int_0^\infty \frac{\exp(-u)}{u^{\alpha/2}} du \right\}, \quad (14) $$

in (12) is the c.f. of an elementary univariate gamma p.d.f. $9(uk; k + m, \frac{1}{2} \alpha)$. Therefore (12) gives an expression for the joint c.f. of exchangeable gamma random variables as a series of products of elementary univariate gamma c.f.'s.

A combination of (12) and (14) yields an expression for the multivariate p.d.f. of exchangeable gamma random variables, which is

$$ f_{\alpha}(u) = f_{\alpha_1, \ldots, \alpha_N}(u_1, \ldots, u_N), \quad (8) $$

111. C.F. OF COMBINER OUTPUT SNR

In this section, we first derive expressions for the joint c.f. and p.d.f. of the branch SNR's. We then apply the p.d.f. expression and the exchangeability property of the branch SNR's to obtain a formula for the c.f. of the combiner output SNR as a series of elementary c.f.'s.

The c.f. in (7) can be rewritten as

$$ \psi_{\alpha}(\beta) = \left\{ \det(\beta) \right\}^{-m} \left\{ \det\left(K^{-1} - 2\text{diag}(\beta)\right) \right\}^{-m}. \quad (9) $$
where \( g(\cdot, \star) \) is given by (13). This expression is a series of products of elementary univariate gamma p.d.f.'s.

Let \( y_1, \ldots, y_N \) denote the instantaneous SNR's in descending order, that is, \(-|y_1| > |y_2| > \cdots > |y_N|\). Using the property of exchangeability, the joint p.d.f. of this ordered set of instantaneous SNR's is given by

\[
J_{N(y_1, \ldots, y_N)}(u_1, \ldots, u_N) = \begin{cases} 
\prod_{i=1}^{N} f_{\gamma_i}(u_i) u_i^{N-1} e^{-u_i \gamma_i}, & \text{if } u_1 > u_2 > \cdots > u_N, \\
0, & \text{otherwise}. 
\end{cases}
\] (16)

The instantaneous SNR of the combiner output for H-S/MRC in which \( L \) branches with largest instantaneous SNR out of \( N \) available branches are chosen and combined using MRC is written as \( \gamma_{H\text{-S/MRC}} \).

The Process Of B/S/mc can be viewed as a Special case of generalized diversity combining (GDC), that was proposed in [4, 5], for which the instantaneous SNR of the combiner output is expressed as

\[
\gamma_G = \sum_{k=1}^{L} \alpha_k \gamma_k, \quad \alpha_1, \ldots, \alpha_L \geq 0. 
\] (17)

From (16), the c.f. of \( \gamma_G \) can be written as

\[
\Psi_{\gamma_G}(w) = \prod_{i=1}^{N} \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} f_{\gamma_i}(u_i) u_i^{N-1} e^{-u_i \gamma_i} du_1 \cdots du_N. 
\] (18)

To evaluate the \( N \)-fold integral in (1e), we look at (15) and (13) and focus on the integral

\[
\int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} f_{\gamma_i}(u_i) u_i^{N-1} e^{-u_i \gamma_i} du_1 \cdots du_N. 
\] (19)

This is evaluated by using the result

\[
\int_{0}^{\infty} u^{a-1} e^{-u} du = \frac{1}{a}, \quad a > 0, \quad p = 0, 1, 2, \ldots. 
\] (20)

repeatedly in (19).

From (15), (18), and (19), we finally obtain

\[
\Psi_{\gamma_G}(w) = \left( \frac{1}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \sum_{\text{all } \alpha_k \geq 0} \left( \frac{\alpha_1 \cdots \alpha_L}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \prod_{k=1}^{L} \left( \frac{1}{\Gamma(\gamma_k)} \right) 
\times \sum_{\text{all } \alpha_k \geq 0} \left( \frac{\alpha_1 \cdots \alpha_L}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \prod_{k=1}^{L} \left( \frac{1}{\Gamma(\gamma_k)} \right) \Phi_{\gamma_1, \ldots, \gamma_N}(w), 
\] (21a)

where

\[
\Phi_{\gamma_1, \ldots, \gamma_N}(w) = \left( \frac{1}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \sum_{\text{all } \alpha_k \geq 0} \left( \frac{\alpha_1 \cdots \alpha_L}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \prod_{k=1}^{L} \left( \frac{1}{\Gamma(\gamma_k)} \right) \times \sum_{\text{all } \alpha_k \geq 0} \left( \frac{\alpha_1 \cdots \alpha_L}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \prod_{k=1}^{L} \left( \frac{1}{\Gamma(\gamma_k)} \right). 
\] (21b)

Note that each term \( \frac{1}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \) is a valid c.f. Thus the above two equations (21a) and (21b) represent an expression for the c.f. of the instantaneous combiner output SNR in the case of GDC as a series of elementary c.f.'s. Substituting \( \gamma_{H\text{-S/MRC}} \) for \( \gamma_G \) in (21b) is a valid c.f. Thus the above two equations (21a) and (21b) represent an expression for the c.f. of the instantaneous combiner output SNR in the case of GDC as a series of elementary c.f.'s. Substituting

\[
\gamma_{H\text{-S/MRC}} = \sum_{k=1}^{L} \alpha_k \gamma_k, \quad \alpha_1 = \ldots = \alpha_L = 1, \quad \alpha_{L+1} = \ldots = \alpha_N = 0
\]

in \( * \ldots (w) \) gives the c.f. \( * \ldots (w) \) of \( \gamma_{H\text{-S/MRC}} \).

IV. SYMBOL ERROR PROBABILITY EVALUATION

The c.f. expression (21) can be conveniently used to obtain the SEP of coherent detection of M-ary modulations. The SEP for H-S/MRC in multipath-fading environments is obtained by averaging the conditional SEP over the channel ensemble as

\[
^*CH\text{-S/MRC} = E^{*CH\text{-S/MRC}} \{ P \{ e|\gamma_{H\text{-S/MRC}} \} \}, 
\] (22)

where \( P \{ e|\gamma_{H\text{-S/MRC}} \} \) is the conditional SEP, conditioned on the random variable \( \gamma_{H\text{-S/MRC}} \).

In the following, we will use coherent MPSK as an example to illustrate the SEP evaluation using the results of Section 11. For coherent detection of MPSK, the technique of [14] gives

\[
P_{k,H\text{-S/MRC}} = \frac{1}{\pi} \int_{0}^{\infty} \Psi_{\gamma_{H\text{-S/MRC}}} \left( -i \gamma \right) d\gamma. 
\] (23)

where

\[
-\text{P S K} = s \left( \frac{n}{M} \right) \text{ and } 0 = a(M - 1)/M. 
\] (24)

From (21), this results in

\[
P_{k,H\text{-S/MRC}} = \left( \frac{1}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \sum_{\text{all } \alpha_k \geq 0} \left( \frac{\alpha_1 \cdots \alpha_L}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \prod_{k=1}^{L} \left( \frac{1}{\Gamma(\gamma_k)} \right) 
\times \sum_{\text{all } \alpha_k \geq 0} \left( \frac{\alpha_1 \cdots \alpha_L}{\Gamma(\gamma_1 + \cdots + \gamma_N)} \right)^L \prod_{k=1}^{L} \left( \frac{1}{\Gamma(\gamma_k)} \right). 
\] (25a)

The notation \( P (\cdot) \) represents the probability.
where

\[ (25b) \]

\[ \mathcal{P}_{n,\ldots, r_n}(\theta) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n+r_n} d\theta, \]  

such that

\[ \mathcal{P}_{n,\ldots, r_n}(\theta) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta, \]  

(26)

We now provide a method to evaluate the integral (25c) in closed form as a canonical expression of the weighted sum of elementary SEP's. The expression in (25c) can be rewritten as

\[ \mathcal{P}_{n,\ldots, r_n}(\theta) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta, \]  

where \( N' = N - L + 1 \), and the \( i_n \)'s are given by

\[ \mu_n = \left\{ \begin{array}{ll} 1 & \text{if } n = 1 \\ i_{n-1} & \text{if } n = 2, \ldots, N'. \end{array} \right. \]  

(27)

\[ \sum_{i=1}^{\text{NUE}} \{ \text{CMPSK} \wedge U \} \hspace{1cm} \text{at} \ U \hspace{1cm} \text{distinct and and} \]

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we can write the integrand in (26) as

\[ f(x) = \prod_{n=1}^{N} \left( \frac{\alpha_n}{\alpha_n + x} \right)^{\mu_n}, \]  

(29)

Using the Heaviside theorem for partial fraction expansion, we obtain the decomposition formula

\[ f(x) = \sum_{n=1}^{N} \sum_{k=1}^{\mu_n} \frac{1}{(\alpha_n - k)^{\mu_n - k} \mu_n!} \left( a^m f(x - k) \right) \]  

where the coefficients \( W_{n,k} \) of the expansion for \( n=1, \ldots, p \) are given by

\[ W_{n,k} = \frac{1}{\alpha_n(k \mu_n - k)^{\mu_n - k} \mu_n!} \left( a^m f(x - k) \right) \]  

(31)

By using Faa di Bruno's formula for the derivatives of a composite function as in [15], we can compute (31) as

\[ W_{n,k} = \left( \prod_{p=1}^{n} \left( \frac{L+1}{m} \right)^{n} \right) \sum_{p=1}^{n} \prod_{p=1}^{n} \left( \frac{L+1}{m} \right)^{n} \]  

(32)

The integral (26) can now be written as

\[ \mathcal{P}_{n,\ldots, r_n}(\theta) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta, \]  

(33)

where \( \text{CMPSK} \wedge U \) is given by another integral

\[ S_n(a_n, b_n) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta. \]  

By using the binomial theorem, this can be shown to be equivalent to

\[ S_n(a_n, b_n) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta, \]  

(34a)

where

\[ T_n(a_n, b_n) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta, \]  

(34b)

By changing the variable of integration to \( t^* \), we get

\[ T_n(a_n, b_n) = \frac{1}{\pi} \int_0^{\pi} \left[ \frac{\sin^2 \theta}{\text{CMPSK} \left( \frac{L+1}{m} \right) + \sin^2 \theta} \right]^{n} d\theta, \]  

(34c)

Equation (33) is in the form of a canonical expression as a weighted sum of the elementary SEP's.

V. NUMERICAL RESULTS

The SEP of QPSK \( (M=4) \) has been numerically computed using (25) together with (33), (32), and (34) and
the resdta plotted in Figs. 1 and 2. The SEP has been calculated using a truncated series version of (25a) with a relative error tolerance of 0.05. In this truncation process, a maximum of 4 term in the summation over $p$ in (25a) \((0 \leq p \leq 3)\) has been required.

Fig. 1 shows the SEP versus the correlation coefficient $p$. MRC Corresponds to $L = N$ while selection mmbining corresponds to $L = 1$. The SEP increases with increase of $p$, as expected. The SEP as a function of $r$ is shown in Figs. 2(a) and 2(b). We find that for given fading param-\text{meter} m and diversity order N, selecting more branches by increasing $L$ improves the performance, but the improve-\text{ment} diminishes with increase of $L$.

REFERENCEX


