Modified Airy Function Solutions to Optical Waveguide Problems

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ABSTRACT

Modified Airy Function (MAF) method gives an approximate solution of the wave equation for planar waveguides with an arbitrary refractive index distribution. Unlike WKB method, the MAF solution does not diverge at the turning points and there is no need of connection formulae. Comparison with exact results for practical waveguides show that the errors in MAF solutions are small. Use of I order perturbation theory together with the MAF solution reduces very much the error in the eigenvalues. It has been shown that an improvement of the method gives almost exact eigenvalues and the eigenfunctions. Though the method will be illustrated with examples of optical wave-guides, it is equally applicable to problems in Quantum Mechanics and other areas of Physics and Engineering.

It is an approximate method [1] to solve a second order linear differential equation:

\[ \Psi''(x) + \Gamma^2(x)\Psi(x) = 0 \]  

where prime denotes differentiation with respect to \( x \) and \( T^2(x) \) is an arbitrary but known function of \( x \) except for a possible eigenvalue constant. The method is applicable to both the initial value as well as the eigenvalue problems. In optical waveguide theory, for a medium characterized by a refractive index distribution \( n''(x) \), the \( y \)-component of the electric field can be written in the form

\[ E_y(x,y,z,t) = H(x)\exp[i(\cot \gamma - \lambda t)] \]

where \( H(x) \) satisfies Eq. (1) with \( T^2(x) = k^2n^2(x) - \lambda^2 \).

Thus

\[ V''(x) + (k^2n^2(x) - \lambda^2)M(x) = 0 \]  

where \( k_0 = \omega/c \) and \( 3 \) is the propagation constant. Equation (2) describes the TE modes of a slab waveguide. The use of WKB method is well established in problems of quantum mechanics and optical waveguides. Following the WKB methodology we assume [2] a solution of Eq. (1) of the form

\[ \Psi(x) = F(x)Ai(x) \]  

or

\[ G(x)Bi(x) \]  

where \( Ai(x) \) and \( Bi(x) \) are solutions of the Airy equation:

\[ y''(x) - xy(x) = 0 \]

Substitution of Eq. (3) in Eq. (1) gives

\[ F''(x)Ai''(x) + 2F'(x)Ai'(x)\xi'(x) + F(x)Ai''(x)\xi(x) - Ai'(x)F(x)\xi'(x)\xi''(x) + T^2(x) = 0 \]  

where prime denotes differentiation with respect to the argument. Eq. (4) is rigorously correct. We choose \( E_x(x) \) so that

\[ \xi[\xi'(x)]^2 + \Gamma^2(x) = 0 \]

the solution of which gives [2]

\[ \xi(x) = \left\{ \frac{3}{2} \int \sqrt{-\Gamma^2(x)dx} \right\}^{2/3} \]

\( \gamma \), here, is the turning point where \( T^2(x) \) has a zero of order one. If we neglect the term proportional to \( F'' \) in Eq. (4) (that is the only approximation we will make), it becomes

\[ 2F'(x)Ai''(x) + F(x)Ai'(x)\xi'(x) = 0 \]

the solution of which is...
The approximate solution that we will discuss, then, is

$$\Psi = \frac{\text{const.}}{\sqrt{\xi(x)}}$$  \hspace{1cm} (7)

where $C_i$ and $C_2$ are constants to be determined from the initial or boundary conditions. Before illustrating the method by a few examples from optical waveguides let us rewrite Eq. (2) in terms of the dimensionless variables in the following form

$$\frac{d^2\Psi}{dX^2} + V^2 \left[ N^2(X) - B \right] \Psi(X) = 0,$$  \hspace{1cm} (9)

where $X = x/d$ ($d$ is a suitably chosen length, $V$ is the normalized waveguide parameter ($n_c$ is the maximum refractive index of the core and $n_2$ the index of the cladding), $\alpha = \frac{p^2 |k| - n}{n_2 - n_1}$ is the normalized propagation constant and $N(X) = \frac{n^2(X) - n_2^2}{n_1 - n_2}$.

Example 1. Consider a planar optical waveguide with refractive index variation given by (see inset of Fig. 1)

$$n^2(x) = n_1 + (n_2 - n_1) \exp(-x/d) \quad \text{for } x > 0,$$

$$n^2(x) = n_2^2 \quad \text{for } x < 0.$$  \hspace{1cm} (10)

where $d$ is the diffusion depth of the waveguide. In this case Eq. (9) becomes

$$\frac{d^2\Psi}{dX^2} + V^2 \left[ \exp(-X) - B \right] \Psi(x) = 0 \quad \text{for } X > 0,$$

$$\frac{d^2\Psi}{dX^2} - V^2 (B + B_i(x)) \Psi(x) = 0 \quad \text{for } X < 0,$$  \hspace{1cm} (11)

where $B = \frac{n_1 n_2^2}{n_1^2 - n_2^2}$.

For $X < 0$, Eq. (10) has an exact solution which is given by

$$\Psi(X) = \exp \left( \frac{x}{\sqrt{V}} \right).$$  \hspace{1cm} (12)

For $X > 0$, the MAF (see Eq. (8)) solution can be written as [3-5]

$$\Psi(X) = \left( \frac{\xi_0^*}{\xi_0} \right)^{1/2} \left( \frac{\xi_0}{\xi_0^*} \right)^{1/2} \Psi_0(X).$$  \hspace{1cm} (11)

Subscript 0 indicates the value of the function at $X = 0$. It may be noticed that we have neglected the solution proportional to $R(q)$ as it will diverge for large $X$. The proportionality constants in Eqs. (11) and (12) have been so chosen as to satisfy the continuity of $\Psi(X)$ at $X = 0$ with $\Psi'(0) = 1$. The eigenvalue equation can be written by using the continuity of $\frac{\Psi(X)}{\xi_0^*}$ at $X = 0$ with $\Psi'(0) = 1$.

We used $n_i = 2.177$, $n_o = 1$ and $n^2 - n_i = 0.187$ in our calculations and compared the MAF results with exact [6] and those calculated by the WKB method [2]. For $V = 1.5$, $b_{\text{exact}} = 0.03501$, $b_{\text{MAF}} = 0.03609$ and $b_{\text{WKB}} = 0.03783$. The error is only 3% (error in WKB is 8%).
Figure 1 shows $\Psi_{MAF}$ versus $X$ for $V = 4$ as calculated by Eq. (12) and $\Psi_{EXACT}$ [6]. The figure shows no discernable difference between the two curves even at the turning point.

**Symmetric Profile:** The method becomes particularly simple if the refractive index profile is symmetric i.e. if

$$r^2(\ast) = r^2(-\ast) \tag{13}$$

For a bound mode, $\gamma(x)$ must tend to $0$ as $X \to \pm \infty$, so we again reject the solution proportional to $Bi$ function. Hence

$$\Psi(x) = \frac{const.}{\sqrt{\gamma(x)}} Ai[\xi(x)] \tag{14}$$

Equations (2) and (13) suggest that the solutions are either symmetric or antisymmetric in $x$, so $\gamma'(0) = 0$ for symmetric solutions and $\gamma'(0) = 0$ for antisymmetric solutions. This leads to the eigenvalue equations

$$Ai'[\xi(0)] \frac{1}{\xi'(0) \xi(0)} = 0 \quad \text{(symmetric solution)} \tag{15}$$

$$A\tilde{\xi}'(0) = 0 \quad \text{(antisymmetric solution)} \tag{16}$$

Let $-Z_n (n = 1, 2, 3, ...)$ denote the nth zero of the $Ai$ function i.e.

$$Ai(-Z_n) = 0, \tag{16}$$

Using Eqs. (6), (15) and (16) and assuming $F^2(x)$ to be positive from $0$ to $x_0$, we obtain

$$\xi(0) = -\left[\frac{3}{2} \int_0^x \Gamma(x) dx\right]^{1/2} \tag{17}$$

or

$$\int_0^x \Gamma(x) dx = (\xi_{\infty} + 1/2)\pi / 2, \tag{17}$$

where

$$r = \sqrt[3]{\frac{4}{3h} 7^{3/2} - 1/2}. \tag{17}$$

Equation (17) is similar to the WKB quantisation condition [2,7] except that (for the WKB case) $E_{\infty}$ is replaced by odd integers. Similarly it can be shown that for symmetric solutions
\[ \int_0^1 \Gamma(x) \, dx = \left( \xi_m + \frac{1}{2} \right) \pi / 2, \]  

(18)

where \( \xi_m \) is the \( m \)th zero of the function \( xA_i'(x) + Ai(x) \), \( Z_m \) and \( C_m \) are universal constants (see Ref. 2, Table 6.2).

Example 2, Let us consider a symmetric refractive index profile (see Fig. 2(a)) given by

\[ n^2(x) = n_0^2 + (n_f - n_i) \exp(-X/X_0) \]

(19)

In terms of the dimensionless variables the equation to be solved (see Eq. (9)) is

\[ V(x) + r^2(x) \xi'(x) = 0 \quad \text{with} \quad r^2(x) = V^2 \left[ \exp(-\Lambda X) \right] - b. \]

The solution of which is given by Eqs. (14) and (6) and the eigenvalue for the first antisymmetric mode can be determined from the following relation (see Eq. (17))

\[ \left( \int_0^1 r(x) \, dx \right) \{C_m + U2 \} \pi / 2 \quad \text{with} \quad \xi_m = 1.01734 \quad \text{[see Ref. 2].} \]

We used \( n_0 = 2.177 \), \( n_f - n_l = 0.187 \), \( X_0 = 632.8 \text{ nm} \) and \( d = 0.4236 \text{ um} \) in our calculations. It gives \( b_{\text{MAF}} = 0.056578 \). Corresponding value of \( b_{\text{EXACT}} = 0.055643 \) and \( b_{\text{WKB}} = 0.05929. \) The error by MAF method is 1.7% compared to an error of 7% with WKB. Figure 2(b) shows the variation of \( \psi(X) \) with \( X \). Again it can be observed that while the WKB solution diverges at the turning points the MAF solution matches very well with the exact.

**Perturbation Correction:** The fact that the MAF eigenfunction matches very well with the exact wavefunction, can be used to improve the eigenvalues with the help of the first order perturbation theory. It can be shown by simple substitution that the MAF solution given by

\[ \Psi(X) = \text{const.} \frac{\sqrt{\xi'(X)}}{\sqrt{F^2(X)}} A_i(\xi(X)) \]  

(20)

is an exact solution of the following differential equation

\[ \frac{d^2 \Psi}{dX^2} + \frac{\xi'(X)Y}{\sqrt{F^2(X)}} \xi'(X) \left( \frac{1}{2} \xi'^2 - \frac{3}{4} \xi'^2 \right) \Psi(X) = 0 \]  

(21)

where \( F^2(X) = V^2 \left[ N_i'(X) - b \right] \) in problems of optical waveguides. Comparing Eqs. (21) and (9) and considering the last term in Eq. (21) as a perturbation, we get a first order correction \( b \) to the normalized propagation constant [5,7].
If we apply the above perturbation correction to both the examples given above we get

Example 1: \( b_{\text{MAF+Per}1} = 0.03502 \) (comparison with exact shows an error of only \( \sim 0.03\% \)).

Example 2: \( b_{\text{MAF+Per}2} = 0.055656 \) (comparison with exact shows an error of only \( \sim 0.024\% \)).

Thus the application of the 1st order perturbation theory greatly improves the eigenvalues but the eigenfunctions remain the same.

**Improved MAF:** In the MAF method described so far we have neglected the term proportional to \( F^2(x) \) (see Eq. (4)). If we retain this term then in place of Eq. (5) we will get

\[
\xi(x)\xi'' + \Gamma^2(x) = -\frac{F(x)}{F(x)}.
\]

Using Eq. (7) we get

\[
\xi'' + \Gamma^2(x) = -\left[\frac{3}{4} \left(\frac{\xi''}{\xi}\right)^2 - \frac{1}{2} \frac{\xi''(x)}{2 r W}\right].
\]

For an arbitrary \( F^2(x) \), it is not easy to find the solution of \( \Psi(x) \). Using \( \xi(x) \) found by the MAF method (see Eq. (6)) one can [8] determine the RHS of Eq. (23). Let us denote it by \( f(x) \). Thus

where \( y'(x) = F(\xi) - f(x) \), is a known function of \( x \) except for an eigenvalue constant in \( F^2(x) \). Equation (24) can now be solved the same way as Eq. (5) to obtain an improved value of \( \xi \) (let us denote it by \( \xi' \)) given by

\[
\xi'(x) = \left\{\frac{3}{2} \left[\int \sqrt{y''(x)} dx\right]\right\}^{2/3},
\]

where \( x_c \) is the turning point of \( y''(x) \) i.e. \( y''(x_c) = 0 \). The solution is now given (see Eq. (8)) by

\[
\Psi(x) = \frac{1}{\sqrt{d\xi'/dx}} \left[C_1 Ai(\xi'(x)) + C_2 Bi(\xi'(x))\right],
\]

\( C_1, C_2 \) and \( b \) are obtained by using the boundary conditions.

**Example 3.** Let us consider a truncated parabolic refractive index profile given by

\[
n'(x) = n' \left[n'^2 - n^2\right] \quad \text{for} \ 0 < x < a; \quad n(x) = n_c \quad \text{for} \ x < 0 \quad \text{and} \quad n(x) = \infty \quad \text{for} \ x > a.
\]

The normalized frequency \( V \) is defined as

\[
V = k_o A \sqrt{n_c^2 - n_a^2}
\]

The values of the various parameters used in the calculations [8] are: \( n_c = 2.177, n = 1 \) and \( n_a^2 - n_c^2 = 0.187 \) and \( V = A \). The value of the normalized propagation constant obtained from the improved MAF method (denoted by \( b_{\text{MAF}} \)) is 0.32703. One can compare it with other results: Exact value of \( b = 0.32617; b_{\text{KB}} = 0.30790 \) and \( b_{\text{MAF}} = 0.31691 \).
The error in $\psi(x)$ is plotted as a function of $X$ in Fig. 3. It can be noticed that by the improved method the maximum error in the wavefunction is reduced to about one fifth of the corresponding error by the MAF method.

With the above improvement, it becomes an extremely accurate method to obtain the eigenvalues as well as the eigenfunctions for planar waveguides with arbitrary index profile. Thus the method should be very useful in the better designing of many optical waveguide based devices and components.

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References