Analysis of Rake Reception with Multiple Symbol Weight Estimation for Antipodal Signaling

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Abstract — In a rake receiver for coherent binary antipodal signaling with weight estimation by matched filtering using the reference signal along with the decisions of the previous $M$ symbol intervals, and predetection maximal-ratio combining (MRC), the estimation errors are not independent of the additive noise, and do not fit into the Gaussian weighting error model for MRC. Here we analyze the error performance of the receiver by: (1) obtaining the conditional symbol error probability (SEP), conditioned on past decisions, from the characteristic function of the decision variable, (2) getting the unconditional SEP using a Markov model of the decision process.

I. INTRODUCTION

For wideband signals, such as CDMA signals, received over a fading channel, a rake receiver resolves the multipaths via code correlation and then combines them [1, 2]. Predetection maximal-ratio combining (MRC) is the preferred combining technique when the reception is coherent, since it gives the maximum instantaneous signal-to-noise ratio (SNR) at the combiner output. However, MRC requires estimation of the diversity branch gains, scaled versions of which are the tap weights of the rake receiver. Errors arising from this estimation process degrade the rake receiver performance. Analysis of MRC with Gaussian distributed estimation errors, which are independent of the additive noise in the channel, has been done in [3]. However, in a rake receiver for coherent binary antipodal signaling with (1) a delayed received signal configuration, (2) tap weight estimation by matched filtering using the reference signal along with the decisions of the previous $M$ symbol intervals, and (3) predetection MRC (similar to that given in [4, p. 803, Fig. 14-5-5] for $M = 1$), the estimation errors are not independent of the additive noise, and do not fit into the model of [3]. This calls for a different approach to the analysis of the rake receiver, and is the focus of this paper.

II. SYMBOL ERROR PERFORMANCE

Consider a diversity reception system over a fading channel with $L$ branches, which uses a rake receiver. When symbol $q_i$, $q_i \in \{-1,1\}$, is transmitted in the $i$th symbol interval of duration $T$, the complex baseband received signal can be represented as

$$ r(t) = \sum_{i=1}^{L} n_s(t - iT) \left( s_i(t) + n(t) \right), \quad iT \leq t < (i+1)T, \quad (1) $$

where $s_q(t)$ is the complex baseband information-bearing signal corresponding to symbol $q_i$, having support $[0, T]$, average symbol energy $2E_s$, and large bandwidth $\gamma$ satisfying $W < \gamma \leq L$, $g_l$ is the random complex gain of the $l$th branch, and $n(t)$ representing the additive noise, is a zero-mean complex circular white Gaussian random process with two-sided power spectral density $2N_0$. The fading is assumed to be frequency-selective but time-flat. The branch gains $\{g_l\}$ are assumed to be invariant over $M+1$ consecutive symbol intervals and are independent of $n(t)$.

Let $s_1(t) = -s_2(t) = s(t)$. The signaling waveform $s(t)$ is a wideband signal generated by a pseudorandom sequence, and satisfies

$$ \int_0^{T} s(t - w)^k \ast s(t - \frac{t}{T}) \, dt = \begin{cases} 2E_s & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases} \quad (2) $$

where $(\ast)^*$ denotes the complex conjugate.

In the $i$th symbol interval, the rake receiver computes tap weights $h_1,\ldots, h^M$ that are scaled estimates of the branch gains $g_1,\ldots, g_L$ respectively as in [4, p. 803, Fig. 14-5-5] for $M = 1$, and uses these weights to combine the received signal along with its $L-1$ delayed versions by MRC. The decision variable $D_i$ resulting from coherent reception and MRC can be expressed as

$$ D_i = \text{Re} \left\{ \sum_{l=1}^{L} h^*_l \int_{iT}^{(i+1)T} r(t - iT) \left( s(t - iT)^* - \frac{2\pi}{W} \right) \, dt \right\}. \quad (3) $$

The receiver makes the decision $D_i \geq 0, -1$.

The estimated weights $h^*, h^L, H_{L1}, \ldots$, obtained by matched filtering, are given by

$$ h_{k,i} = \sum_{m=1}^{M} \text{sgn}(D_{i-m}) \int_{iT}^{(i+1)IT} r(t-mT) \left( s(t-mT)^* - \frac{2\pi}{W} \right) \, dt, \quad (4) $$

where $s(t-mT, -\frac{2\pi}{W})$ is the reference signal and $\text{sgn}(\cdot)$ denotes the signum or sign function. Note that the integration in (4) corresponds to a low pass filtering operation. Thus, in the $i$th symbol interval, the tap weights
are estimated by making use of the decision variables -D_{j-i}, 1 \leq i \leq M of the M previous symbol intervals.

Let the random variable N_{k,i} be defined as

\[ N_{k,i} = \sum_{m=1}^{M} \text{sgn}(D_{i-m})\left[2E_s q_{i-m}g_k + N_{k,i-m}\right]. \]  

Similarly, using (3) and (5), the decision variable D_{i} is expressed in terms of A^j, \ldots, N_{m} as

\[ D_{i} = \text{Re}\left\{ \sum_{k=1}^{L} h_{k,i} \left[2E_s q_{i-k}g_k + N_{k,i-k}\right] \right\}. \]

Defining the random variable Y_{k,i} as

\[ Y_{k,i} = \sum_{m=1}^{M} \text{sgn}(D_{i-m})\left[\frac{Y_{k,i-m}+Y_{k,i-n}Y_{k,i-n}}{2}\right], \]

we can rewrite \( D_{i} \) in terms of \( Y_{k,i} \) as

\[ D_{i} = \sum_{k=1}^{L} Y_{k,i} \left[2E_s q_{i-k}g_k + N_{k,i-k}\right]. \]

Note that each \( Y_{k,i} \), conditioned on \( g_{i} \), is a complex circular Gaussian random variable having a \( CN(2E_s q_{i}g_k, 4E_s N_0) \) distribution.

The ideal instantaneous SNR at the combiner's output, denoted as \( \gamma_{\text{tot}} \), is expressed as

\[ \gamma_{\text{tot}} = \sum_{k=1}^{L} \gamma_{k,i} \]

where \( j \) is the number of times \( \langle f_j \rangle \text{sgn}(D_{i-m}) \) takes the value 1, or the number of correct decisions between the \( (i-M) \)th and \( (i-1) \)th symbol intervals.

Let the c.f. of \( \gamma_{\text{tot}} \) be denoted as \( \Psi_{\gamma_{\text{tot}}} \). Averaging \( \Psi_{\gamma_{\text{tot}}} \) in (9a) over the probability density function of \( \gamma_{\text{tot}} \) yields the c.f. of \( D_{i} \), which depends on \( q_{i} \) and \( f_{i} \), and is expressed as

\[ \Psi_{D_{i}}(j\omega, q_{i}, f_{i}) = \frac{\omega}{(1 + 4M\omega^2 E_s^2 N_0^2)} \]

where \( f_{i} = \sum_{m=1}^{M} q_{i-m} \text{sgn}(D_{i-m}) \).

Using properties of Hermitian quadratic forms in complex Gaussian variables [5], we can show that the characteristic function (c.f.) of \( D_{i} \), conditioned on \( \gamma_{\text{tot}} \), is

\[ \Psi_{D_{i}}(j\omega, q_{i}, f_{i}) = \frac{\omega}{(1 + 4M\omega^2 E_s^2 N_0^2)} \]

and \( j = \sqrt{-1} \). Note that since \( \langle f_j \rangle \text{sgn}(D_{i-m}) \in \{-1,1\} \), \( f_{i} \) takes values from \(-M\) to \(M\) at intervals of 2, that is,

\[ f_{i} \in \{-2j-M, \ldots, 2j-M\}, \quad j = 0, \ldots, M. \]

III. PERFORMANCE IN I.I.D. RAYLEIGH FADEING

Consider a Rayleigh fading channel with i.i.d. zero-mean circular Gaussian branch gains \( g_{1}, \ldots, g_{L} \), each with \( E|g_k|^2 = 0 \) and variance \( E|g_k|^2 = N_0 \). Let

\[ \Gamma = \frac{E_s}{N_0} \]

de note the ideal average SNR per branch. The c.f. of \( \gamma_{\text{tot}} \) is given by [4]

\[ \Psi_{\gamma_{\text{tot}}} \]
Substituting (17) in (11), we obtain

\[
\mathcal{D}_z \left( \frac{1}{z^\alpha} \right) = \frac{(-1)^L}{z(z+\alpha_1)^2(z-\alpha_1)^2(M+M\Gamma+f_1^2)^L},
\]

where

\[
\alpha_i = \sqrt{\left( \frac{21}{r+1} \right) + f_1^2 + f_1^2}, \quad b_i = \frac{(1+r)(M+M\Gamma+f_1^2)^L}{M+M\Gamma+f_1^2}.
\]

Note that \( a_i \) and \( b_i \) are both positive quantities. It is clear from (18) that \( \Psi \mathcal{D}_z \left( \frac{1}{z^\alpha} \right) \) has only one pole of order \( L \) at \( z = -a_i \) on the left-half \( z \)-plane, while \( \Psi \mathcal{D}_z \left( \frac{1}{z^\alpha} \right) \) has only one pole of order \( L \) at \( z = a_i \) on the right-half \( z \)-plane. Therefore, substitution of (18) in (13) yields

\[
Pe_i(f_1, f_t) = \frac{1}{2 - 2^L} 
\]

By changing the variable of differentiation in (20b) from \( z \) to \( z' = -z \), we find that

\[
Pe_i(f_1, f_t) = \frac{1}{2 - 2^L} 
\]

By using Faa di Bruno's formula \([7]\) to evaluate the \((L-1)\)th derivative as in \([8]\), we obtain from (20) and (21) the expression

\[
Pe_i(f_1) = 1 - Pe_i(-f_1).
\]

For any symbol interval, the subscript \( j \) of \( p_j \) is the number of correct decisions made over the past \( M \) symbol intervals. Thus \( p_j \) is the conditional SEP in any symbol interval, conditioned on the fact that \( j \) correct decisions have occurred over the past \( M \) symbol intervals. Substituting (19) in (22), we can rewrite \( p_j \) as

\[
p_j = \frac{(M+Mr+(2j-M))r^L}{(2^L/(1+r)(M+(2j-M))^L)} \times \frac{1}{\sqrt{(1+r)(M+(2j-M))^L+(2j-M)^L}} \times \mathcal{E} \left( \frac{L-1}{n(2^L/(1+r)(M+(2j-M))^L)} \right) \right)^{L-1}
\]

We have from (21) and (23) the relation

\[
P(\mathcal{M}j) = I - P_j, \quad j = 0, \ldots, M
\]

which implies that \( p_M = 0 \) when \( M \) is even.

### IV. UNCONDITIONAL ERROR PROBABILITY

Consider the \( M \times 1 \) decision vector \( D_q \), defined as

\[
D_q \equiv \begin{bmatrix}
\frac{1+q_1\text{sgn}(D_1)}{2} \\
\frac{1+q_1\text{sgn}(D_{t-1})}{2} \\
\vdots \\
\frac{1+q_1\text{sgn}(D_{t-M}))}{2}
\end{bmatrix}
\]

which constitutes the decision of the \( t \)th symbol interval \( iT < t < (i+1)T \), and the decisions of the past \( M-1 \) symbol intervals. The entries of \( D_q \) are either zeroes or ones, zero implying an erroneous decision, and one a correct decision. We treat the vector \( D_q \) as a binary number, the first row entry representing the most significant bit, and the \( M \)th row entry the least significant bit. The decimal equivalent of \( D_q \), which we denote as \( Q \), is therefore given by

\[
Q = \sum_{m=1}^{M} \frac{1+q_1\text{sgn}(D_1)}{2^m} \approx 2M - m
\]

where \( a_i \) and \( b_i \) are given by (19). Thus (22) is a closed-form expression for the conditional SEP in i.i.d. Rayleigh fading.

Note that the conditional SEP \( Pe_i(f_i) \) depends on the value of \( f_i \) which ranges from \(-M\) to \( M \) at intervals of 2 (given by (10)), but has no direct dependence on \( i \). We therefore define a probability \( p_j \) as

\[
p_j = Pe_i(2j - M), \quad j = 0, \ldots, M
\]

Note that that different values of \( p_j \) can map onto the same value of \( f_i+1 \). However, for every value of \( Q \), there
is a unique value of \( f_{i+1} \). For example, when \( M = 2 \), we have \( c_i \in \{ 0, 1, 2, 3 \} \), while \( f_{i+1} \in \{ -2, 0, 2 \} \). In this case, \( c_i = 0 \) maps onto \( f_{i+1} = -2 \), \( c_i = 1, 2 \) map onto \( f_{i+1} = 0 \), and \( c_i = 3 \) maps onto \( f_{i+1} = 2 \).

A reasonable measure of the unconditional error probability (UEP) is the probability that at least one error occurs in a block of \( M \) consecutive symbols. The UEP at the \( i \)th symbol interval can be expressed as

\[
P_{ue,i} = 1 - \Pr(c_i = 2^M - 1) = 1 - \Pr(f_{i+1} = M). \tag{29}
\]

We proceed to find a way of obtaining the UEP.

If in the \((i-1)\)th symbol interval, \( c_{i-1} = c \), then in the \( i \)th symbol interval, we have \( c_i = \lfloor \frac{c}{2} \rfloor \) in case of an error, and \( c_i = 2^M - 1 + \lfloor |c| \rfloor \) in case of a correct decision. Let \( C_{i-1} = c \) map onto \( f_i = 2j - M \). We can then express the conditional probability of \( c_i \) given \( c_{i-1} \) as

\[
\Pr(c_i = \lfloor \frac{c_{i-1}}{2} \rfloor | c_{i-1} = c) = \Pr(\text{sgn}(D_i) < 0 | f_i = 2j - M) = \rho_j, \tag{30}
\]

\[
\Pr(c_i = 2^M - 1 + \lfloor |c| \rfloor | c_{i-1} = c) = \Pr(\text{sgn}(D_i) > 0 | f_i = 2j - M) = 1 - \rho_j,
\]

where \( \rho_j \) is given by (24). It is clear that the random variable \( c_i \), which takes integer values from 0 to \( 2^M - 1 \), depends only on the immediate past variable \( C_{i-1} \), and not on the earlier variables. In addition, the conditional probability of \( c_i \) given \( C_{i-1} \) does not depend on \( i \). As a result, the random sequence \( \{ C_{i} \} \) is a discrete-time discrete-state homogeneous Markov process, having the set \( \{ 0, 1, 2, \ldots, 2^M - 1 \} \) of \( 2^M \) states.

Let \( P_S(i) \) denote the state probability vector in the \( i \)th symbol interval, given by

\[
P_S(i) = \begin{bmatrix} P_{S_0}(i) \\ \vdots \\ P_{S_{2^M-1}}(i) \end{bmatrix} = \begin{bmatrix} \Pr(c_i = 0) \\ \vdots \\ \Pr(c_i = 2^M - 1) \end{bmatrix}. \tag{31}
\]

The \( 2^M \times 2^M \) transition probability matrix, denoted as \( P_T \), with rows indexed by states of \( c_i \) and columns by states of \( C_{i-1} \) in the order \( 0, 1, 2, \ldots, 2^M - 1 \), can be easily constructed using (30). Thus we have the relation

\[
P_S(i) = P_T P_S(i-1) = P_T^i P_S(0), \tag{32}
\]

where \( P_S(0) \) is the initial state probability vector.

We begin operating the receiver by receiving a known training sequence \( q_0, q_1, \ldots, q_{-M+1} \). This implies \( q_i = \text{sgn}(D_i) \) for \( i = -M + 1, \ldots, 0 \). Thus \( \Pr(c_0 = 2^M - 1) = 1 \), implying

\[
P_s(0) = [0, 0, \ldots, 0, 1]^T, \tag{33}
\]

where \((\cdot)^T\) denotes transpose.

From (32) and (33), we can get \( P_S(i) \) in terms of the entries of \( P_T \). The UEP is given by

\[
P_{ue,i} = 1 - P_{S_{2^M-1}}(i), \tag{34}
\]

which is in terms of \( \rho_0, \ldots, \rho_{2^M-1} \).

The unconditional SEP in the \( i \)th symbol interval, which we denote as \( P_i \), is given by

\[
P_i = \sum_{j=0}^{2^M-1} \sum_{k=0}^{M-1} \rho_j P_{S_k}(i-1). \tag{35}
\]

Let \( k_1(j), \ldots, k_{\nu_j}(j) \) be the \( \nu_j \) values of \( C_{i-1} \), in ascending order, which map onto \( f_i = 2j - M \). Then the unconditional SEP in (35) can be rewritten as

\[
P_i = \sum_{j=0}^{2^M-1} \nu_j P_{S_j}(i-1). \tag{36}
\]

From the structure of the transitional probability matrix \( P_T \), we get from (36)

\[
P_i = \prod_{j=0}^{2^M-1} P_{S_j}(i), \tag{37}
\]

It is clear from (34) and (37) that \( P_i \leq P_{ue,i} \).

V. EARLIER MODEL

In [3], the errors in the tap weight estimates \( h_{1,i}, \ldots, h_{L,i} \) are modeled as complex Gaussian random variables. These errors are characterized by the squared correlation between \( h_{k,i} \) and \( g_k \), which is denoted as \( \rho^2 \) and is given by

\[
\rho^2 = \frac{2M_e E[|h_{i,j}|^2 (\sin^2 \theta)^k]}{E[|h_{i,j}|^2]}, \quad k = 1, \ldots, L, \tag{38}
\]

Following the approach in [3], the SEP of antipodal signaling in i.i.d. Rayleigh fading, with an ideal average SNR per branch \( \Gamma \), can be expressed as a weighted sum of integrals and is given by

\[
P_e,Gaus(\rho^2) = \sum_{k=1}^{L} A(k) \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{\sin^2 \theta}{\Gamma^2 + \sin^2 \theta} \right)^k d\theta. \tag{39a}
\]

where the weighting coefficient \( A(k) \) can be expressed as

\[
A(k) = \binom{L-1}{k-1} (1 - \rho^2)^{L-k} \rho^{2(k-1)}. \tag{39b}
\]

VI. NUMERICAL RESULTS AND CONCLUSIONS

In Figs. 1 and 2, plots of the unconditional SEP \( P_i \) versus \( P \) resulting from weight estimation by matched filtering are compared with those of the SEP \( P_e,Gaus(\rho^2) \) versus \( P \) resulting from the Gaussian distributed weighting error.
model of [3]. The fading is assumed to be i.i.d. Rayleigh. We have computed $P_i$ using (37) and $P_e, Gaus(\rho^2)$ using (39). The value of \( i \) indicates the number of symbols that are received between two successive training phases. \( P_i \) increases with increase of \( i \), while \( P_e, Gaus(\rho^2) \) decreases with increase of \( \rho^2 \), as expected. For a given \( \rho^2 \), we can find some value of \( i \) for which weight estimation by matched filtering outperforms the Gaussian distributed weighting error model above some value of \( \Gamma \). For example, with \( \rho^2 = 0.5 \) and \( i = 50 \), \( P_i \) is lower than \( P_e, Gaus(\rho^2) \) for \( \Gamma > 6.4 \) dB when \( M = 1 \), \( iL = 8 \) (Fig. 1) and for \( \Gamma > 4.2 \) dB when \( M = 2 \), \( L = 8 \) (Fig. 2). We also observe that an increase in the estimation block size \( M \) from 1 to 2 causes the performance to improve considerably. For example, at \( \Gamma = 9 \) dB, the unconditional SEP for \( M = 1 \), \( i = 50 \) is \( 10^{-6.1} \) while that for \( M = 2 \), \( i = 50 \) is \( 10^{-6.1} \), implying an improvement by a factor of 10. When \( \rho^2 = 1 \), we have MRC with perfect weight estimation, and this gives the best performance.

A plot of the steady state unconditional SEP versus \( \Gamma \) in i.i.d. Rayleigh fading for \( L = 8,10 \) and \( M = 3,...,6 \) is shown in Fig. 3. The steady state unconditional SEP decreases with increase of \( \Gamma \), \( L \), or \( M \); however, the amount of decrease in the steady state unconditional SEP with increase of \( M \) diminishes as \( M \) rises.

REFERENCES


