DYNAMIC RESPONSE OF TRUSSED BRIDGES
FOR MOVING LOADS

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Abstract—A continuum approach is presented to obtain the transverse vibration of trussed bridges traversed by a single moving load. The efficiency and the accuracy of the method are determined by comparing its results with those obtained by the dynamic analysis of the bridge as a discrete lumped mass system, which can account for both truss action and flexural action of the deck in the response. Using the proposed method, a parametric study is performed to show the influence of some important parameters on the dynamic response of the bridge. The parameters include relative stiffness of the bridge deck and truss, number of panels, type of truss and speed parameter.

NOTATION

\( A_{nr}, B_{nr}, \ldots \) integration constants in mode shape expression
\( \{C\}_r \) integration constant matrix for \( r \)th span
\( \text{DAF} \) dynamic amplification factor
\( E \) Young's modulus of elasticity of bridge material
\( \{F\}_r \) end force matrix for \( r \)th span
\( g \) gravitational acceleration
\( l, l_i \) moment of inertia of beam
\( \{K\}_r \) dynamic stiffness matrix of \( r \)th span
\( \{K\} \) global dynamic stiffness matrix
\( \{\bar{K}\} \) global dynamic stiffness matrix after superimposing the spring stiffness matrix
\( L \) span of beam
\( L_t \) total length of bridge
\( m \) average mass per unit length of bridge
\( p(x, t) \) external load on beam
\( P \) magnitude of concentrated load
\( \{R\}_r \) integration constant matrix for \( r \)th span
\( S_f \) spring factor
\( t, t_i \) time instant
\( \{U\} \) global unknown end-displacement matrix
\( v \) velocity of concentrated load
\( W(x, t) \) dynamic deflection function
\( \{X\}_r \) end force matrix for \( r \)th span
\( \alpha \) speed parameter
\( \phi, \phi_t \) mode shape function
\( \zeta, \zeta_0 \) damping ratio
\( \mu, \mu_i \) mass per unit length of beam
\( \omega, \omega_0 \) natural frequency

1. INTRODUCTION

Dynamic behaviour of bridges to moving loads has been a topic of considerable interest in the recent past. The literature on the subject includes the dynamic response of various types of bridges such as multi-span continuous bridges, suspension bridges and cantilever bridges. The types of dynamic loading on the bridge considered as those produced due to speed and braking of the vehicle, surface irregularity of the pavement and initial bounce of the vehicle. Most of these works used the lumped mass approach (with or without rotational inertia assigned to the point masses) [1–3]. However, a few of them also used the continuum approach based on distributed mass and stiffness [4–6]. The first method computes the static stiffness matrix of the structure at the nodes where masses are lumped and computes the natural frequencies and mode shapes through standard eigen solutions. In the second method, a dynamic stiffness matrix is formed which includes both the elastic and inertial forces acting on the structure during vibration, and modal parameters are found through the solution of frequency equations [7, 8]. Another method proposed by Melosh and Smith [9] assumes a simpler displacement field of a rod compared to the exact or finite element model, and finds the exact modes and frequencies for the vibration of skeletal structures such as trusses, frames or beams. Considerable work is also reported on the dynamic behaviour of continuous beams which find the natural frequencies, vibration modes and steady-state forced vibrations of the beam due to a single- or multi-axle moving load, both by the lumped mass [10, 11] and continuum approach [4, 12].

As such, there is not much theoretical work available on the dynamic response of a trussed bridge due to vehicular movement. Recently, Cai et al. [13] presented a transverse vibrational analysis of plane truss using an analytical method which provides a closed form solution to the problem. However, the paper did not address the problem of dynamic response of trusses due to moving loads.

In this paper, a continuum approach is presented for the dynamic analysis of a trussed bridge traversed by a single moving load. Using this method of analysis, a parametric study is conducted to investigate the dynamic behaviour of the trussed bridge under vehicular movement. The advantage of the continuum approach for the dynamic analysis of the trussed bridge lies in the fact that (i) the method of analysis becomes computationally more efficient compared to lumped mass approach, and (ii) it
permits a parametric study to be conducted in terms of certain system parameters which can be effectively varied within a meaningful range of nondimensional numbers. The applicability of the proposed method is demonstrated by comparing its results with those of the analysis based on the usual stiffness approach with lumped masses.

2. ASSUMPTIONS

The following assumptions are made for the formulation of the problem:

(i) the deck is supported on cross girders and bracings which connect the trusses on either side of the bridge at the deck level nodes;
(ii) the load from the deck is transferred to the truss joints by simply supported beam action of the deck between two cross girders;
(iii) the deck with its supporting system is assumed to have an equivalent moment of inertia against bending;
(iv) the moving vehicle is represented by a single unsprung concentrated load traversing along the centreline of the bridge deck so that a two-dimensional (2-D) idealization is possible;
(v) the dynamic effect produced due to surface roughness of the pavement is neglected;
(vi) the mass of the deck including the deck level chord of the truss is assumed to be large compared to the total mass of the other members of the truss.

3. THEORY

With the above idealizations, the 2-D model of the trussed bridge is shown in Fig. 1. The deflection of the deck between two nodes as shown in Fig. 1 is influenced by the stiffness of the truss and simply supported deck stiffness between the two nodes as outlined in assumption (iii). The dynamic response of the system to a traversing load can be obtained using the usual stiffness approach with lumped masses at the dynamic degrees of freedom. This method of analysis is briefly described before the proposed continuum approach.

3.1. Stiffness method of analysis with lumped masses

The idealized trussed bridge shown in Fig. 1 is considered as a multi-degree of freedom with lumped mass system as shown in Fig. 2. While two dynamic degrees of freedom are considered at the top nodes, only one (vertical) dynamic degree of freedom is considered for the nodes along the deck. In order to accurately consider the effect of bending action on the dynamic response of the deck, a sufficient number of in-between nodes (or mass points) are to be considered along the deck between two joints of the truss.

With the lumped masses considered at each node corresponding to the translational degrees of freedom as described above, the problem is converted into a multi-degree of freedom lumped mass system subjected to a vertical concentrated moving load along the deck. The problem can be solved by numerical integration using any standard algorithm, such as Newmark's beta method [14]. The efficiency of the solution can be enhanced by using normal mode theory. The main disadvantages of the method are the computational time and the storage requirement.

3.2. Continuum approach

Since the dynamic response of the deck is of main interest, the equivalent continuum model for the 2-D trussed bridge system (Fig. 2) can be idealized as shown in Fig. 3(a). The stiffness of the truss corresponding to the vertical degrees of freedom at the deck level joints is provided by the stiffness of an interconnected spring system. The spring system has the same coupled stiffness matrix as the condensed stiffness matrix of the truss corresponding to the vertical degrees of freedom at the deck level. At each node, total mass of an appropriate portion of the truss is lumped. The mass per unit length and the flexural rigidity of the beam between two nodes correspond to those of the half-width of the deck between the two cross girders. In order to use continuum approach, lumped masses at the nodes are distributed over a small length as shown in Fig. 3(b).

Over this small length, moment of inertia of the equivalent continuum is taken to be very small for simulating a hinged condition at the nodes.

The governing equation of motion for any segment \( r \) of the continuum (beam) (neglecting shear

![Fig. 1. Trussed bridge](image1)

![Fig. 2. Two-dimensional lumped mass model for trussed bridge](image2)
where \([K]r\) is called the 'dynamic stiffness matrix' for the \(r\)th segment; \([F]r\) and \((X)r\) are the end-force and displacement vectors. The integration constants \(Anr, Bnr, \text{ etc.}\), are related to the end-displacements as

\[
(C)r = [R]r (X)r,
\]

where \([R]r\) is called the 'integration constant matrix' for the \(r\)th span. The procedure for determining the elements of the matrices \([K]r\) and \([R]r\) is explained in Appendix I where the elements are also explicitly given.

Assembling the individual dynamic stiffness matrices for the segments \(r (r = 1, 2, \ldots, N)\), after discarding the rows and columns corresponding to the restrained degrees of freedom, the global dynamic stiffness matrix \([K]\) for the beam is formed.

The coupled spring stiffness is then properly superimposed on \([K]\) to obtain the final dynamic stiffness matrix \([\bar{K}]\). The condition for the free vibration of the beam may then be written as

\[
[R] (U) = \{0\},
\]

where \((U)\) is the unknown end-displacement vector for the beam corresponding to the dynamic degrees of freedom. Equation (6) leads to

\[
\det[R] = 0.
\]

Using the regula falsi approach\([15]\) the natural frequencies for the system are determined with certain accuracy from the solution of eqn (7). Once the natural frequencies are obtained, mode shapes can be known through the use of eqns (6), (5) and (2).

3.3. Calculation of dynamic response

Using mode superposition technique, the solution of the equation of motion eqn (1) can be written as

\[
W(x, t) = \sum_{n=1}^{\infty} \phi_n(x) T_n s(t),
\]

where \([W](x, t)\) is the deflection of the \(r\)th span at \(x\) and at time \(t\), when the load is on the \(r\)th span; \(T_n s(t)\) is the displacement function in generalized coordinates for the \(r\)th term of the series and \(\phi_n\) has already been defined by eqn (2). Note that \(t_s\) is measured from the instant when the load is at \(x = 0\) and \(r = s\).

Expanding the load in a similar series

\[
p(x, t_s) = \sum_{n=1}^{\infty} q_n(t_s) \phi_n(x),
\]

where

\[
q_n(t_s) = \frac{P}{Mn^2} \phi_n(C),
\]
and
\[ M_n^2 = \int_0^L \phi_n^2(x) \, dx \]  
(11)
\[ = \sum_{r=1}^N \int_0^L \phi_n(x_r) \, dx_r, \]  
(12)

where \( L \) is the length of \( r \)th segment and \( l \) is the summation of lengths of all segments. Substituting eqns (8) and (9) into eqn (1) and using eqn (10) we obtain
\[ \ddot{T}_n(t_r) + 2\rho \dot{T}_n(t_r) + \omega_n^2 T_n(t_r) = \tau_n \phi_n(t_r), \]  
(13)
where \( \rho = \xi \omega_n \), \( \xi \), being the modal damping ratio, \( \tau_n = P/(\mu MN^2) \), and the overdot denotes derivative with respect to time \( t \). Using the Laplace transform technique, the solution of eqn (13) can be expressed as
\[ T_n(t_r) = \Phi_n(t_r) + \theta_n(t_r) \]  
(14)
in which the expression for \( \Phi_n(t_r) \) is given in Appendix 2, and
\[ \theta_n(t_r) = T_n(0) \exp(-\rho t_r) \cos(\omega_n t_r) \]  
\[ + \frac{1}{\omega_n} \left\{ \rho T_n(0) + \ddot{T}_n(0) \right\} \]  
\[ \times \exp(-\rho t_r) \sin(\omega_n t_r), \]  
(15)
where \( \omega_n = \sqrt{\omega_n^2 - \rho^2} \), \( \omega_n \gg \rho \), \( T_n(0) \), \( \ddot{T}_n(0) \) are integration constants to be evaluated from initial conditions at \( t_r = 0 \), which can be expressed by the following recurrent relations
\[ T_n(0) = \Phi_n(t_r = l_{r-1} \omega_n) + \theta_n(t_r = l_{r-1} \omega_n) \]  
\[ \ddot{T}_n(0) = \dot{\Phi}_n(t_r = l_{r-1} \omega_n) + \dot{\theta}_n(t_r = l_{r-1} \omega_n). \]  
(16)

4. PERFORMANCE OF THE PROPOSED METHOD

To verify the continuum method, the trussed bridge shown in Fig. 1 is solved both by the stiffness approach with lumped masses and the proposed continuum approach. The following properties for the bridge are considered in the numerical example: area of bottom chord of the truss = 160.0 cm²; equivalent flexural rigidity of the deck = 17,028.55 kN m²; \( E = 1.96 \times 10^6 \) kN/m²; \( \xi = 0 \), height of truss = 5.0 m; span = 30.0 m; length of each panel = 50.0 m; mass/unit length of half-width deck = 800.0 kN/m; areas of diagonals, top chord and verticals are 0.8, 0.6 and 0.3 times that of the bottom chord; masses per unit length of diagonals, top chord and verticals are, respectively, 0.12, 0.15 and 0.1 times that of the half-width deck. A single moving load of 49.0 kN moves at a uniform speed of 20.0 m/sec over the bridge deck.

Table 1 shows the first ten natural frequencies obtained by the two methods. They are in close agreement.

Mode shapes and frequencies are also determined by the continuum approach with the lumped mass of the truss at the joints along the deck instead of being distributed over the small lengths adjacent to the joints as shown in Fig. 3(b). The concentrated masses in the continuum approach are considered by subtracting the quantities \( m_i \omega_i^2 \) from the diagonal elements of assembled dynamic stiffness matrix, where \( m_i \) is the \( i \)th concentrated mass lumped at the \( i \)th joint. The results are also shown in Table 1 (listed under Method 3). It is seen that the results obtained by Method 3 are nearly same as those obtained by the other two methods.

The dynamic amplification factors (DAF) for deflection at the middle of the fourth panel of the deck are also shown in Table 1. The results are practically the same for the three methods. Thus, Method 3, which does not strictly satisfy the conditions of orthogonality of mode shapes for continuum solution [16], can be used for all practical purposes. The advantage of the method is that it drastically reduces the dynamic degrees of freedom compared to the other continuum method (Method 2), and therefore, saves considerable computational time.

<table>
<thead>
<tr>
<th></th>
<th>Lumped mass approach</th>
<th>Distributed masses over small lengths</th>
<th>Concentrated masses at nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Eigenvalues</strong></td>
<td><strong>Method 1</strong></td>
<td><strong>Method 2</strong></td>
<td><strong>Method 3</strong></td>
</tr>
<tr>
<td>Eigenvalues 39.68</td>
<td>38.92</td>
<td>38.87</td>
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<td>56.69</td>
<td>54.56</td>
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<td>106.80</td>
<td>103.98</td>
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<td>187.65</td>
<td>188.49</td>
<td>187.53</td>
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<tr>
<td>206.60</td>
<td>202.01</td>
<td>202.57</td>
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<tr>
<td>211.60</td>
<td>208.46</td>
<td>208.84</td>
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<tr>
<td>232.20</td>
<td>233.80</td>
<td>233.43</td>
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</table>

<table>
<thead>
<tr>
<th><strong>DAF</strong></th>
<th><strong>Method 1</strong></th>
<th><strong>Method 2</strong></th>
<th><strong>Method 3</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>DAF 12.40</td>
<td>11.68</td>
<td>12.77</td>
<td></td>
</tr>
</tbody>
</table>
5. PARAMETRIC STUDIES

To study the dynamic behaviour of trussed bridge, a nondimensional ‘spring factor’, $S_f$ defining the relative stiffnesses between the truss and the deck is introduced as

\[ S_f = \frac{E I_{deck}}{S_n} \]

where $E I_{deck}$ is the rigidity of any reference panel of deck between two truss nodes; $I$ is the length of the reference panel; $S_n$ is the diagonal element of the coupled spring stiffness matrix (mentioned earlier) corresponding to a reference spring $(i)$.

**Effect of the parameter $S_f$.** The trussed bridge of Fig. 4(a) is analysed to show the effect of the spring factor on the dynamic behaviour of the bridge. The height of the truss and the length of each panel of deck are considered as 6.9 and 6.0 m, respectively; the data regarding the properties of deck and truss members are same as the previous problem. To define the spring factor, the reference spring is taken at the central node of the deck. Further to this, the spring factor is varied by changing the $E I$ value of the deck.

The variation of DAF for deflection at the middle of fourth panel with $S_f$ is shown in Fig. 5. DAF generally increases with the increase in $S_f$.

**Effect of number of panels.** The Warren trussed bridges shown in Fig 4(a–c) are analysed to show the influence of the number of panels on the DAF of the bridge deck. For all the cases, the total mass of the half-width bridge, areas of the members of the truss and $E I / l^3$ of the deck are kept the same, which are 85,000.0 kg, 160.0 cm$^2$ and 2.60 $\times$ 10$^5$ N/m, respectively. Other properties are the same as the first example. Since the number of panels are changed for the three cases, the size of the coupled stiffness matrix representing the spring stiffness matrix for the continuum solution is changed. Also, the spring factor (defined for $S_n$ at the central joint of the deck for all the cases) becomes different. The values of the spring factor for the three cases are computed as 2.743 $\times$ 10$^{-2}$, 2.532 $\times$ 10$^{-3}$ and 1.266 $\times$ 10$^{-3}$. Because of these changes, the values of the DAF at the midspan of the bridge differ for the three cases, for example, they are 21.11%, 41.79% and 23.89%, respectively. Note that the DAF does not necessarily increase with the decrease in the number of panels. In fact, the results show that the DAF could be same for both a large and small number of panels. At the same time, results indicate that the number of panels can have significant influence on the DAF.

**Effect of truss type.** To study this effect three kinds of trussed bridges are analysed as shown in Fig. 6(a–c). The span, panel length, height, areas of the members, total mass of the truss and deck rigidity are kept the same for all the three cases. The mass of half-width deck and $E I / l^3$ for all the panels is 90,000.0 kg and 2.65 $\times$ 10$^5$ N/m, respectively. Other properties are the same as the first example. The DAFs for the deflection at the midspan of the deck for the three cases are obtained at 32.46%, 22.43% and 61.50%, respectively. The difference in results is mainly due to the difference in the coupled spring stiffness matrix of the system for the three cases.

**Effect of speed parameter.** Speed parameter ($\alpha$) is defined by the relation

\[ \alpha = \sqrt{\left( \delta / g \right) / 2L} \]

in which

\[ \delta = \left( L m g \right) L^2 / 48 E I, \]

where $m$ and $I$ are average values of mass/unit length and moment of inertia of deck, respectively; $L$ is
the length of the bridge, $g$ is the acceleration due to gravity and $v$ is the speed of the vehicle.

The trussed bridge as shown in Fig. 7 is analysed for different values of $a$. Area of top chord and mass of half-width deck including the top chord are considered as 300.0 cm$^2$ and 1000.0 kg/m, respectively; area of bottom chord, diagonals and verticals are 0.8, 0.4 and 0.3 times that of the top chord, respectively; mass of those members are 0.2, 0.15 and 0.1 times that of the half-width deck, respectively; $S_i = 0.01$; $\xi = 0$. The variation of DAF for deflection at the middle of the fifth panel of the deck with $a$ is shown in Fig. 8. The variation of DAF with $a$ does not show any consistent pattern. It, however, indicates that the speed of the vehicle has a significant effect on the DAF and increase in the speed does not necessarily increase the DAF.

Response of a three-span bridge with continuous truss system. A three-span continuous trussed bridge (Fig. 9) is analysed to determine the DAF for deflection at the middle of the intermediate span for three different spring factors. The spring factor is defined for the reference spring at the central node of the bridge deck. The DAFs are shown in Table 2 where the same are also shown for the trussed bridge (with same member properties) if it were discontinuous at the supports.

6. CONCLUSIONS

A continuum approach is presented for the dynamic analysis of simple or multi-span trussed bridges traversed by a single moving load. The eigenvalues of the bridge and the DAF for deflection due to the traversing load are compared with those obtained from the usual stiffness method with lumped masses. The proposed method is found to be comparatively much faster than the usual lumped mass approach. The following conclusions can be drawn from the present study:

(i) The eigenvalues and DAFs computed from the lumped mass approach and the two continuum approaches are practically same.
(ii) The value of DAF generally increases with the increase in the spring factor.
(iii) The number of panels in the truss has a significant effect on the DAF.
(iv) The type of truss (keeping the weight, number of panels, span and member properties same) has significant influence on the DAF.

<table>
<thead>
<tr>
<th>Spring factor for continuous bridge</th>
<th>DAF for continuous truss system (%)</th>
<th>Corresponding DAF for discontinuous truss system (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0016</td>
<td>21.15</td>
<td>16.34</td>
</tr>
<tr>
<td>0.0020</td>
<td>15.42</td>
<td>5.44</td>
</tr>
<tr>
<td>0.0024</td>
<td>31.22</td>
<td>9.72</td>
</tr>
</tbody>
</table>
(v) The variation of DAF with the speed parameter does not follow a consistent pattern; increase in the speed of the vehicle may not necessarily increase the DAF.

(vi) The DAF for the case of a multi-span continuous trussed bridge is generally higher than for the same trussed bridge discontinuous over the supports.

REFERENCES


APPENDIX 1. FORMULATION OF DYNAMIC STIFFNESS MATRIX

Depending on the number of dynamic degrees of freedom there can be four types of dynamic stiffness matrices.

(a) Moments present at both ends

Using eqn (2) and expressions for the end displacements \((X)r\) and end forces \((F)r\) at the two ends of any beam segment \(r\), both the end displacements and forces can be expressed in terms of integration constants as

\[
\{X\}r = [A]r[C]r
\]

and

\[
\{F\}r = [B]r[C]r,
\]

where \([A]r\), \([B]r\) are \(4 \times 4\) matrices and \([C]r\) is the vector containing constants \(\alpha_{er}\), etc. Expressing \([C]r\) in terms of \([X]r\) from eqn (17) and substituting this into eqn (18),

\[
\]

Also, from eqn (17), \([C]r\) may be written as

\[
\]

The elements of the \(4 \times 4\) matrices \([K]r\) (symmetric) and \([R]r\) may be expressed as follows:

\[
\begin{align*}
\kappa_{11} &= G(\sin \lambda \cosh \lambda + \cos \lambda \sinh \lambda), \\
\kappa_{12} &= -G \sin \lambda \sinh \lambda / \beta, \\
\kappa_{13} &= -G(\sin \lambda + \sinh \lambda), \\
\kappa_{14} &= G(\cos \lambda - \cosh \lambda) / \beta, \\
\kappa_{22} &= G(\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda) / \beta, \\
\kappa_{23} &= -\kappa_{14}, \\
\kappa_{24} &= -G(\sin \lambda - \sinh \lambda) / \beta, \\
\kappa_{33} &= \kappa_{11}, \\
\kappa_{34} &= -\kappa_{12}, \\
\kappa_{44} &= \kappa_{12},
\end{align*}
\]

where

\[
\begin{align*}
\beta &= \beta_m, \\
\lambda &= \beta l, \\
G &= EI \beta^2 / (1 - \cos \lambda \cosh \lambda),
\end{align*}
\]

and

\[
\begin{align*}
r_{11} &= -H(1 + \sin \lambda \sinh \lambda - \cos \lambda \cosh \lambda), \\
r_{12} &= H(\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda) / \beta, \\
r_{13} &= -H(\cos \lambda - \cosh \lambda), \\
r_{14} &= -H(\sin \lambda - \sinh \lambda) / \beta, \\
r_{22} &= H(\sin \lambda \cosh \lambda + \cos \lambda \sinh \lambda), \\
r_{23} &= H(1 - \sin \lambda \sinh \lambda - \cos \lambda \cosh \lambda) / \beta, \\
r_{24} &= -H(\sin \lambda + \sinh \lambda), \\
r_{33} &= -r_{11} / \beta, \\
r_{34} &= -r_{12} \beta, \\
r_{44} &= -r_{12} \beta, \\
r_{41} &= -r_{13}, \\
r_{42} &= -r_{13}, \\
r_{43} &= -r_{14}, \\
r_{44} &= -r_{14} / \beta,
\end{align*}
\]

where \(H = 1 / 2(1 - \cos \lambda \cosh \lambda)\).
(b) Moment zero at the right-hand end

Eliminating the constant \(Dw\) [see eqn (2)] using the zero moment condition, and following the procedure as in case (a), the elements of the \((3 \times 3)\) \([K]\)\(r\) matrix and \((4 \times 3)\) \([R]\)\(r\) matrix are obtained as

\[
\begin{align*}
k_{11} &= 2S \cos \lambda \cosh \lambda, \\
k_{12} &= -S(\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda) / \beta, \\
k_{13} &= -S(\cos \lambda + \cosh \lambda), \\
k_{21} &= 2S \sin \lambda \sinh \lambda / \beta^2, \\
k_{22} &= S(\sin \lambda + \sinh \lambda) / \beta, \\
k_{23} &= S(\sin \lambda \cosh \lambda), \\
k_{31} &= S(1 + \cos \lambda \cosh \lambda), \\
k_{32} &= -T(1 + \cos \lambda \cosh \lambda + \sin \lambda \sinh \lambda) / 2, \\
k_{33} &= -T \cosh \lambda, \\
k_{41} &= -T \sinh \lambda / \beta, \\
k_{42} &= -T \sinh \lambda / \beta, \\
r_{11} &= -\frac{1}{2}, \\
r_{12} &= 0, \\
r_{13} &= 0, \\
r_{21} &= -\frac{1}{2}, \\
r_{22} &= 0, \\
r_{23} &= 0, \\
r_{31} &= -\frac{1}{2}, \\
r_{32} &= 0, \\
r_{33} &= 0, \\
r_{41} &= T(-\lambda + \sin \lambda \sinh \lambda - \cos \lambda \cosh \lambda) / 2, \\
r_{42} &= T \cos \lambda, \\
r_{43} &= T \sin \lambda / \beta.
\end{align*}
\]

where

\[
S = El, \beta^3 / (\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda),
\]

and

\[
\begin{align*}
T &= 1 / (\sin \lambda \cosh \lambda - \cos \lambda \sinh \lambda).
\end{align*}
\]

(c) Moment zero at the left-hand end

Similarly, eliminating \(Cw\) [see eqn (2)] by using zero moment condition, the elements of the two matrices are expressed as

\[
\begin{align*}
k_{11} &= S(1 + \cos \lambda \cosh \lambda), \\
k_{12} &= -S(\cos \lambda + \cosh \lambda), \\
k_{13} &= -S(\sin \lambda + \sinh \lambda) / \beta, \\
k_{21} &= 2S \cos \lambda \cosh \lambda, \\
k_{22} &= S(\cos \lambda \sinh \lambda + \sin \lambda \cosh \lambda) / \beta, \\
k_{23} &= S(\cos \lambda \cosh \lambda), \\
k_{31} &= S(1 + \cos \lambda \cosh \lambda), \\
k_{32} &= -T(1 + \cos \lambda \cosh \lambda + \sin \lambda \sinh \lambda) / 2, \\
k_{33} &= -T \cosh \lambda, \\
k_{41} &= -T \sinh \lambda / \beta, \\
k_{42} &= -T \sinh \lambda / \beta,
\end{align*}
\]

where

\[
Q = El, \beta^3 / (2 \sin \lambda \sinh \lambda),
\]

and

\[
\begin{align*}
r_{11} &= -\frac{1}{2}, \\
r_{12} &= 0, \\
r_{13} &= 0, \\
r_{21} &= -\frac{1}{2}, \\
r_{22} &= 0, \\
r_{23} &= 0, \\
r_{31} &= -\frac{1}{2}, \\
r_{32} &= 0, \\
r_{33} &= 0, \\
r_{41} &= -\frac{1}{2}, \\
r_{42} &= -\frac{1}{2},
\end{align*}
\]

APPENDIX 2

Referring to eqn (14) the expression for \(\Phi_a(t)\) is given by

\[
\Phi_a(t) = \frac{r}{a_0} \int_0^t \phi_a(\omega \tau) \exp[-\rho(t + \tau)] \sin \omega(t + \tau) \, d\tau.
\]

The definite integral can be split into four parts as given below

\[
\Phi_a(t) = \frac{r}{a_0} (t_1 + t_2 + t_3 + t_4),
\]

where

\[
I_1 = \text{An} \pi(X_1 + X_2 - X_3 - X_4) / 2
\]

in which

\[
\begin{align*}
X_1 &= \{ -\rho \sin \sigma t + (\sigma + \epsilon) \cos \sigma t \} / d, \\
X_2 &= \{ \rho \sin \sigma t - (\sigma - \epsilon) \cos \sigma t \} / d, \\
X_3 &= \{ \rho \sin \sigma t + (\sigma + \epsilon) \cos \sigma t \} \exp(-\rho t) / d, \\
X_4 &= \{ \rho \sin \sigma t - (\sigma - \epsilon) \cos \sigma t \} \exp(-\rho t) / d.
\end{align*}
\]
\[ X_4 = \left[ \rho \sin \epsilon t - (\sigma - \epsilon)\cos \epsilon t \right] \exp(-\rho t)/d_2 \]
\[ \sigma = \beta \mu \nu , \]
\[ \epsilon = \alpha , \]
\[ t = t , \]
\[ d_1 = \rho^2 + (\sigma + \epsilon)^2 , \]
\[ d_2 = \rho^2 + (\sigma - \epsilon)^2 , \]
\[ I_2 = \text{Dns}(Y_1 - Y_2 - Y_3 + Y_4)/2 \]

in which

\[ Y_1 = \left[ \rho \cos \epsilon t + (\sigma + \epsilon)\sin \epsilon t \right]/d_1 \]
\[ Y_2 = \left[ \rho \cos \epsilon t + (\sigma - \epsilon)\sin \epsilon t \right]/d_2 \]
\[ Y_3 = \left[ \rho \cos \epsilon t - (\sigma + \epsilon)\sin \epsilon t \right] \exp(-\rho t)/d_1 \]
\[ Y_4 = \left[ \rho \cos \epsilon t + (\sigma - \epsilon)\sin \epsilon t \right] \exp(-\rho t)/d_2 \]
\[ I_3 = \text{Cns}(Z_1 + Z_2 - Z_3 - Z_4)/2 \]

in which

\[ Z_1 = \epsilon \exp(\sigma t)/d_1 \]
\[ Z_2 = \epsilon \exp(-\sigma t)/d_4 \]
\[ Z_3 = [\epsilon \cos \epsilon t + (\rho + \sigma)\sin \epsilon t] \exp(-\rho t)/d_1 \]
\[ Z_4 = [\epsilon \cos \epsilon t + (\rho - \sigma)\sin \epsilon t] \exp(-\rho t)/d_4 , \]

where

\[ d_3 = \epsilon^2 + (\rho + \sigma)^2 , \]
\[ d_4 = \epsilon^2 + (\rho - \sigma)^2 \]

and

\[ I_4 = \text{Dns}(Z_1 - Z_2 - Z_3 + Z_4)/2 . \]