VIBRATION OF CONTINUOUS BRIDGES UNDER MOVING VEHICLES

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The dynamic behaviour of multi-span continuous bridges under a moving vehicular load is investigated by considering the effect of interaction between the vehicle and the bridge pavement, the torsion in the bridge due to eccentrically placed vehicles and the randomness of the surface irregularity of the pavement. The response of the bridge is obtained in the time domain by using an iterative procedure employed at each time step to take into account the non-linearity of the pavement–vehicle interactive force. The solution procedure is made efficient by utilizing a continuum approach for determining the eigenfunctions of the bridge deck, and by obtaining the deck response at each iteration with the help of a few closed form (recursive) expressions. The method of analysis is used to perform a parametric study to show the influence of some important parameters on the response of the bridge.

1. INTRODUCTION

The determination of dynamic response of bridges due to the passage of moving vehicles is a problem of considerable interest. In design practice, the static load is increased by an impact factor to account for the dynamic effect of the load. The impact factor is usually specified by formulae based on the length of the bridge in the bridge codes. However, this method of design does not include the effects of various factors such as the type of bridge, speed, initial bounce, mass distribution, suspension characteristics of the vehicle and the profile of the bridge surface.

Veletzos and Huang [1] are among the few early researchers who introduced the bridge–vehicle interaction in the dynamic response of the bridge. They considered a two-dimensional multi-axle tractor–trailer vehicle model and a lumped mass model of the bridge, and solved the bridge equation of motion by numerical integration, using a modified moment distribution method for finding the influence coefficients. Prior to their work, quite a few studies were made of the problem of the vibration of simple and continuous beams under moving loads. These studies are available in the textbook by Fryba [2]. The Veletzos and Huang [1] vehicle model, with or without modifications, was generally employed to obtain the dynamic response of bridges by many authors. Using this model, Gupta and Traill-Nash [3, 4] investigated the effects of braking and initial bounce of the vehicle for a single-span bridge. They idealized the bridge as (i) an orthotropic plate and (ii) a beam (with and without torsional freedom) with lumped masses. Mulcahy [5] utilized an orthotropic plate model of a single-span bridge to obtain the dynamic response of a three-axle tractor–trailer vehicle, and considered the effects of braking and eccentric placement of the vehicle, and the bridge surface roughness. He considered a surface profile under each tyre strip approximated by a quadratic parabola.

rough pavement on the bridge-vehicle dynamic system was considered by Iabanathan and Wieland [7]. They simulated the vehicle-pavement interactive force from a specified power spectrum of this force (for a particular vehicle moving at a particular speed) to analyze a single-span bridge. Snyder and Wormley [8] considered the dynamic interaction between a two-axle vehicle and an elevated guideway idealized as a series of simple-span beams. The effect of random guideway roughness was considered by generating a roughness profile from the specified power spectral density function of the roughness, with the help of a digital recursive filter. A direct numerical integration technique was employed to solve the vehicle and guideway equations of motion simultaneously. Recently, Coussy et al. [9] utilized a canonical representation for the random bridge surface and a specified power spectrum of the surface to analyze a single-span bridge for the first three modes.

In the present study, the response of a continuous bridge traversed by a vehicle modelled as both a single sprung and an unsprung mass is obtained by modelling the bridge deck as a continuum system. The pavement surface irregularity is modelled as a stationary random process characterized by a specified power spectral density function. The response analysis is carried out in the time domain by simulating the random profile of the bridge deck. The effects of the torsion due to the eccentric placement of the vehicle and the non-linearity produced due to vehicle-pavement interaction are duly considered in the analysis. Finally, a parametric study is conducted to show the influence of some important factors on the mean response of the bridge.

2. ASSUMPTIONS

The following assumptions are made for the formulation of the problem: (i) the bridge deck is treated as a continuous Bernoulli-Euler beam and assumed to have a linear elastic behaviour; (ii) torsional rotations of the bridge at the supports are assumed to be zero; (iii) warping and distortion effects in the torsion of the bridge deck are small enough to be neglected; (iv) the vehicle is idealized as a single sprung mass system.

The suspension characteristics of the vehicle form a bilinear force-deformation relationship for the idealized vehicle [1].

3. CONTINUUM APPROACH FOR BRIDGE DECK ANALYSIS

The governing equations of motions for transverse and torsional vibration for any beam (idealized deck) segment (see Figure 1), with shear deformation and rotatory inertia neglected, are

\[ EI \partial^4 W(x, t) / \partial x^4 + 2 \xi \omega m \partial W(x, t) / \partial t + \ddot{m} \partial^2 W(x, t) / \partial t^2 = p(x, t) \]  
\[ GJ \partial^2 \theta(x, t) / \partial x^2 - I_\theta \partial^2 \theta(x, t) / \partial t^2 = m_\theta(x, t), \]

![Figure 1. Continuum model of bridge.](image-url)
in which $EI$, $GJ$, $m$, and $I_m$ are the flexural rigidity, torsional rigidity, mass and mass moment of inertia per unit length of the beam, respectively, $W(x,t)$ and $\theta(x,t)$ are the transverse deflection and torsional rotation functions, $p(x,t)$ and $m_i(x,t)$ are the values of the transverse and torsional load at $x$ at any time $t$, $\xi$ is the damping ratio and $\omega$ is the natural frequency of the beam (a list of symbols is given in Appendix 2).

For an eccentrically placed vehicle (with respect to the line of centre of mass of the bridge), equations (1) and (2) are generally coupled since the right side load intensities $p(x,t)$ and $m_i(x,t)$ depend upon the bridge displacements $W(x,t)$ and $\theta(x,t)$.

For an unsprung mass idealization of the vehicle moving over a smooth bridge pavement, the load intensities may be written as

$$p(x,t) = W(x,t) \delta(x-vt), \quad m_i(x,t) = p(x,t) \epsilon,$$  \hspace{1cm} (3)

where

$$\dot{y}(x,t) = \dot{W}(x,t) + \dot{\theta}(x,t) \epsilon,$$  \hspace{1cm} (4)

and in which $\delta$ denotes the Dirac delta function, $y(x,t)$ is the total deflection of the bridge deck at a transverse distance of $\epsilon$ from the centreline, $\epsilon$ is the eccentricity of the vehicle path from the centreline of the bridge, $v$ is the speed of the vehicle, $W$ is the weight of the vehicle, and $g$ is the gravitational acceleration. Generally, $\dot{W}(x,t)$ and $\dot{\theta}(x,t)$ are very small for bridges, and hence $\dot{y}$ can be neglected. As a result, the load intensities given by equation (3) remain uniform for an unsprung mass system and equations (1) and (2) become uncoupled.

3.1. TRANSVERSE MODES OF VIBRATION

The expression for the $n$th mode shape for transverse vibration of the $r$th beam segment is

$$\phi_n(x_r) = A_n \cos \beta_n x_r + B_n \sin \beta_n x_r + C_n \cosh \beta_n x_r + D_n \sinh \beta_n x_r,$$  \hspace{1cm} (5)

where $A_n$, $B_n$, $C_n$ and $D_n$ are integration constants expressed in terms of the $n$th natural frequency $\omega_n$ and $\beta_n = \lambda_n / \epsilon$ is defined by

$$\beta_n^r = m_n \omega_n^2 EI,$$  \hspace{1cm} (6)

where the suffix $r$ denotes the $r$th segment of the beam and $\lambda_n$ is the eigenvalue of the system to be obtained from the free vibration analysis. The origin of $x$, is taken from the left of the $r$th segment.

By utilizing equation (5), a relation between the end forces and the end displacements for the $r$th segment may be written as

$$\{F\}_r = [K] \{X\}_r,$$  \hspace{1cm} (7)

where $\{F\}_r$ and $\{X\}_r$ are end force and end displacement vectors and $[K]$ is called the "dynamic stiffness matrix" of the $r$th beam segment. The integration constants $A_n$, etc., are related to the end displacements by

$$\{C\}_r = [R] \{X\}_r,$$  \hspace{1cm} (8)

where $\{C\}_r$ is the vector of the integration constants containing $A_n$, etc., and $[R]$ is called the "integration constant matrix". Expressions for the elements of the matrices $[K]$, and $[R]$, can be found in reference [6]. The overall dynamic stiffness matrix is obtained by assembling the stiffness matrix for each segment $r$. The natural frequencies and mode shapes of the system are determined by solving the eigenvalue problem.
3.2. TORSIONAL MODES OF VIBRATION

The expression for the \( n \)th mode shape of the \( r \)th span in torsional vibration is

\[
\Omega_n(x_r) = \tilde{A}_n \cos \tilde{\beta}_n x_r + \tilde{B}_n \sin \tilde{\beta}_n x_r,
\]

where \( \tilde{A}_n \) and \( \tilde{B}_n \) are defined in the same way as in transverse vibration and

\[
\tilde{\beta}_n = I_m \omega_n^2 / GJ_r,
\]

in which \( GJ_r \) and \( I_m \) are the torsional rigidity and mass moment of inertia per unit length of the \( r \)th span of the bridge, respectively, and \( \omega_n \) is the \( n \)th natural frequency in torsional vibration. Expressions for the dynamic stiffness matrix and the integration constants matrix for the torsional vibration can be found in reference [11].

3.3. DYNAMIC RESPONSE FOR A SINGLE MOVING LOAD (UNSPRUNG)

By using a mode superposition technique, the solution of the equation of motion (1) can be written as

\[
W(x_r, t_s) = \sum_{n=1}^{\infty} \phi_n(x_r) T_n(t_s),
\]

where \( W(x_r, t_s) \) is the transverse deflection of the \( r \)th span at \( x_r \) and at time \( t_s \) when the load is on the \( s \)th span; \( T_n(t_s) \) is the displacement function in generalized co-ordinates for the \( n \)th term of the series and \( \phi_n \) has already been defined by equation (5). Note that \( t_s \) is measured from the instant when the load is at \( x_r = 0 \) and \( r = s \).

The load is also expanded in a similar series as

\[
p(x_r, t_s) = \sum_{n=1}^{\infty} q_n(t_s) \phi_n(x_r),
\]

where

\[
q_n(t_s) = \frac{P}{M_n^2} \phi_n(C_s) \quad \text{and} \quad M_n^2 = \int_0^L \phi_n^2(x) \, dx = \sum_{r=1}^{N} \int_0^{l_r} \phi_n^2(x_r) \, dx_r,
\]

in which \( C_s (= v t_s) \) is the distance of the load from the left of the \( s \)th span, \( l_r \) is the length of the \( r \)th span, \( L \) is the summation of the lengths of the \( N \) spans (i.e., \( L \) is the total span length) and \( P \) is the magnitude of the moving load. For the case of torsion, \( P \) is replaced by the torsional moment \( M \) in equation (13).

By substituting equations (11) and (12) into equation (1) and using equation (13), one obtains

\[
\ddot{T}_n(t_s) + 2 \rho \dot{T}_n(t_s) + \omega_n^2 T_n(t_s) = \Gamma_n \phi_n(v t_s),
\]

where \( \rho = \xi_n \omega_n \), \( \xi_n \) being the modal damping ratio, \( \Gamma_n = P / (M_n^2 \omega_n^2) \), and \( (\cdot) \) denotes the derivative with respect to time, \( t_s \). By using a Laplace transformation technique, the solution of equation (15) can be expressed as [6]

\[
T_n(t_s) = \Phi_n(t_s) + \Psi_n(t_s),
\]

where

\[
\Phi_n(t_s) = \frac{\Gamma_n}{\alpha_n} \int_0^{t_s} \phi_n(v \tau) \exp \left[ -\rho(t_s) \right] \sin \alpha_n(t_s - \tau) \, d\tau
\]

and

\[
\Psi_n(t_s) = T_n(0) \exp \left[ -\rho(t_s) \right] \cos \alpha_n(t_s) + \left( 1/\alpha_n \right) \left( \rho T_n(0) + \dot{T}_n(0) \right) \exp \left[ -\rho(t_s) \right] \sin \alpha_n(t_s).
\]

in which \( v \) is the speed of the moving load \( P \), as mentioned earlier.
The definite integral \( \Phi_m(t) \) can be split into four parts

\[
\Phi_m(t) = (\Gamma_n/\alpha_n)(I_1 + I_2 + I_3 + I_4),
\]

in which the expressions for \( I_1 \), etc., are given in Appendix 1, and \( \alpha_n = \sqrt{\omega_n^2 - \rho^2} \), \( \omega_n \geq \rho \), \( T_m(0) \) and \( \dot{T}_m(0) \) are integration constants to be evaluated from initial conditions at \( t = 0 \), which can be expressed in the form of the following recursion relations:

\[
T_m(0) = \Phi_{n-1}(l_{n-1}/v) + \Psi_{n-1}(l_{n-1}/v)
\]
\[
\dot{T}_m(0) = \Phi_{n-1}(l_{n-1}/v) + \dot{\Psi}_{n-1}(l_{n-1}/v).
\]

The torsional rotation due to the torsional moment \( M \) can be determined in a similar way as in the case of transverse vibration. The differences are that \( \phi_n \) should be replaced by \( \Omega_n \) (see equation (9)) and \( \xi_n, \alpha_n, I_n \) and \( \rho \) denote the corresponding quantities for the torsional vibration analysis.

The deflection of the deck across its width at any section is determined by superposing the deflections at time \( t \) obtained from the transverse and torsional vibration analyses.

Note that for the case of a pure torsional mode of vibration of a continuous bridge with supports restrained against torsional rotation, each span will vibrate independently without transmitting any torque to the other spans.

### 3.4. Dynamic Response for Sprung Mass System

For a sprung mass system moving eccentrically with respect to the centreline of the mass of the deck, three equations of motion have to be solved simultaneously. The first two equations represent the dynamic equilibrium for the bridge, and are given by equations (1) and (2). The third equation is for the dynamic equilibrium of the sprung mass system. As mentioned before, the interactive force between the pavement and the sprung mass system depends upon the deck displacement and, hence, the three equations are coupled.

The equation of motion of the vehicle idealized as a single sprung mass system (see Figure 2(a)) can be expressed in terms of the change in interaction force \( (P_m - P_\mu) \), as given by [1]

\[
(W_r/g)\ddot{z} = -(P_m - P_\mu),
\]

### Figure 2. (a) Vehicle idealized as a sprung mass system; (b) force–deformation relationship of the sprung system.
in which $W_e$ is the total weight of the vehicle as defined before, $P_n$ is the instantaneous value of the interaction force between the vehicle and the bridge and $P_n^{\text{m}}$ is the corresponding static value, and $\ddot{z}$ is the vertical acceleration of the mass. The force–deformation relation $(P_n - u)$ is shown in Figure 2(b), where $u$ is the relative displacement between the bridge surface and the sprung mass system. The slope of the two straight lines $oa$ and $ab$ are $k$, and $k_n$, where $k_n$ is given by

$$1/k_n = (1/k_i) + (1/k_i),$$

(22)

in which $k_i$ and $k$, are the tyre and suspension spring stiffness of the vehicle. The relative displacement $u$ is given by

$$u = z - y - r,$$

(23)

where $z$ is indicated in Figure 2(a), $y$ is the transverse deflection of the bridge under the sprung mass from the position of rest (i.e., when there is no vehicle on the bridge) and $r$ is the ordinate of the random profile of the pavement under the sprung mass.

Because of the bilinear nature of the force–deformation relationship of the vehicle, equation (21) is solved in the incremental form as

$$\frac{W}{g} \Delta \ddot{z} = -\Delta P,$$

(24)

where $\Delta \ddot{z}$ and $\Delta P$ are differences in the values of $\ddot{z}$ and $(P_n - P_n^{\text{m}})$ between the two time instants $t$ and $(t + \Delta t)$. Equation (24) is solved by using a standard procedure [12] to obtain the incremental displacement first and then the displacement at time $(t + \Delta t)$. However, the solution of equation (24) requires $\Delta P$ to be defined and, for this purpose, the deck displacement at the current time station $(t + \Delta t)$ must be known. This requires an iterative solution at each time station, with all three equations of motion considered simultaneously.

4. ITERATIVE SOLUTION

The iterative procedure at each time instant is described as follows:

(i) At any time $(t + \Delta t)$, define the longitudinal position of the vehicle. The displacement $z$, velocity $\dot{z}$ and acceleration $\ddot{z}$ of the sprung mass, the interaction force $P_n$, and the deck displacement $y$ under the sprung mass are all known at the previous time instant $t$.

(ii) Give a small increment to $P_n$ at time $t$ by an amount $\delta$, to obtain a trial set of interaction force at time $(t + \Delta t)$: i.e., set $\Delta P = \delta$ ($\delta$ being of the order of $10^{-6}$ or so).

(iii) Calculate the values of $P_n$ at time $(t + \Delta t)$ by adding $\Delta P$ to $P_n$ at time $t$.

(iv) Solve the vehicle incremental equation of motion (24) to find $\Delta \ddot{z}$.

(v) Find $\ddot{z}$, $\dot{z}$ and $z$ at time $(t + \Delta t)$ from the known values of those quantities at time $t$ and from $\Delta \ddot{z}$ obtained from the previous step.

(vi) Solve the bridge equations of motion for the load $P_n$ computed in step (iii) and the moment $M$ (= $P_n \times$ eccentricity) for both transverse and torsional vibration (in the case of eccentric vehicle) and find the value of $y$.

(vii) Compute $u$ from equation (23) by using $z$ and $y$ from the previous steps and $r$ from the synthetically generated bridge pavement profile.

(viii) Compute a new value of $P_n$ at time $(t + \Delta t)$ from the force–deformation relationship of the system as shown in Figure 2(b), corresponding to the computed value of $u$ in step (vii).

(ix) Compare the value of $P_n$ from step (viii) with that from step (iii). If the difference between the two values of the interaction force $P_n$ is greater than a prescribed tolerance $\varepsilon$ ($\approx 10^{-4}$), calculate a new value of $\Delta P$ which is equal to the latest available value of $P_n$. 

minus the corresponding value at time $t$. Then repeat steps (iii) through (ix) until the value of $P_m$ converges.

Once the convergence is obtained for any time instant, equation (11) is utilized to calculate the deck deflection and rotation at any required section of the bridge.

5. RANDOM PROFILE OF THE BRIDGE PAVEMENT

The randomness of the bridge surface roughness is specified by its power spectral density function which is given by

$$S(f) = \begin{cases} \tilde{a} f^{-\beta} & \text{if } f_l < f < f_u, \\ 0 & \text{otherwise} \end{cases}$$

(25)

where $f$ is the spatial frequency of the pavement roughness, $f_l$ and $f_u$ are the lower and upper cut-off frequencies, $\tilde{a}$ is the roughness coefficient and $\beta$ is a constant [9]. From the power spectral density function, a number of random surface profiles are generated with the help of the Monte-Carlo [10] simulation technique. The simulations can be carried for both the space and the time co-ordinates. For the latter, the speed of the vehicle is utilized to transfer the spatial frequency to the time frequency. The time history of the random surface profile $r(t)$ is utilized in the present formulation, as is evident from equation (23).

6. NUMERICAL EXAMPLES AND PARAMETRIC STUDIES

A three-span continuous bridge has been analyzed for the following data: $l_1 = l_2 = 30.0$ m, $I_1 = I_2 = 2.5 \times 10^{10}$ Nm$^2$, $EI_1 = 4.5 \times 10^{10}$ Nm$^2$, $m_1 = m_2 = 5.0 \times 10^3$ kg/m, $\bar{m}_2 = 8.0 \times 10^3$ kg/m, $I_{m1} = I_{m2} = 1.8 \times 10^3$ kg m$^2$, $GJ_1 = GJ_2 = 1.44 \times 10^{10}$ Nm$^2$, $GJ_1 = GJ_2 = 2.88 \times 10^{10}$ Nm$^2$, $\zeta = 0$ for all modes.

It is assumed that the mass of the bridge deck is uniformly distributed, so that the centre of mass for the bridge deck lies along the centreline (longitudinal) of the deck. Furthermore, the supports of the bridge deck provide complete restraint against torsional rotations.

The idealized single mass vehicle data is $W_e = 1.0 \times 10^5$ N and $\mu = 0.15$ with the two spring constants $k_1$ and $k_2$, being varied depending upon the problem.

The two important parameters, namely, the speed parameter and the frequency ratio, used in the numerical study are defined as follows: the speed parameter ($\alpha$) is defined as

$$\alpha = \beta_{11} \nu/\omega_1 = \lambda_{11} \nu/l_1 \omega_1,$$

(26)

where $\omega_1$ is the fundamental frequency of the transverse vibration of the bridge, $\beta_{11}$ corresponds to equation (5), $\lambda_{11} = \beta_{11}/l_1$, and $\nu$ is the speed of the vehicle as mentioned earlier; the frequency ratio ($f_r$) is defined as

$$f_r = \omega_1/\omega_1,$$

(27)

where $\omega_1$ is the natural frequency of the idealized vehicle, defined with respect to the initial (tyre) stiffness of the sprung mass system as

$$\omega_1 = \sqrt{(k_s g/W_e)}.$$

(28)

For the given frequency characteristics of the bridge, the natural frequency of the vehicle $\omega_1$ is automatically defined once the frequency parameter $f_r$ is assumed. Since $k_s$ is related to $\omega_1$ (through equation (28)), it is also defined via $\omega_1$. The value of the other spring constant, $k_i$, is determined by assuming a ratio of $k_i/k_s$. 


The first five natural frequencies (in rad/s) of the transverse and torsional vibration of the bridge are shown in Table I. For the case of torsion, each span vibrates separately because of the restraints at the supports against the torsional rotations. Thus, for five sets of natural frequencies, there will actually be ten independent natural modes of vibration. In each of the ten modes, either one of the end spans or the intermediate span will have mode shape ordinates; for the other two spans, the mode shape ordinates will be zero.

In Figure 3 are shown typical plots of the influence lines for the deflection at the middle of the intermediate span of the bridge when the vehicle is assumed to move along the centreline of the bridge deck with a speed of 15.0 m/s. The bridge profile is assumed to be smooth and the $k_e/k_i$ ratio is considered to be 0.75. The figure shows the influence lines for three cases: (i) when the vehicle is idealized as an unsprung mass; (ii) when the vehicle is idealized as a sprung mass system (Figure 2(a)); and (iii) when the vehicle moves with a crawling speed, i.e., the static influence line for deflection at the above-mentioned point.

The maximum dynamic deflections obtained for the vehicle modelled as unsprung and sprung mass systems are compared in Table 2 for different values of speed parameter and frequency ratio. For a particular speed parameter, the difference between the responses for unsprung and sprung mass models is practically negligible. In Figure 3 these two responses cannot be distinguished from each other. As a result, only two curves are evident in the

![Figure 3. Deflection curves for smooth bridge pavement; $T =$ total time.](image-url)
figure. Thus, the bridge–vehicle interactive force remains practically the same for both cases (i) and (ii). This is so because the time variation of the dynamic deflection of the bridge is such that the acceleration is very small. As a result, the sprung mass system does not undergo significant oscillation with respect to the bridge pavement (i.e., $u$ in equation (23) is very small). This results in a very small fluctuation of the interactive force about its mean static value, as is evident from Figure 2(b). Therefore, for all practical purposes, a vehicle can be idealized as an unsprung mass system for a smooth bridge pavement. When the vehicle moves eccentrically along a distance of 4·0 m away from the bridge centreline ($x = 0·09, f_r = 1·1267$), the maximum deflections of the bridge at a distance of 5·0 m away from the centreline are obtained as 0·004248 m and 0·004320 m for the unsprung and sprung mass systems, respectively. Thus, for the eccentric movement of the vehicle, an unsprung mass idealization of the vehicle may be adopted for a smooth pavement.

As the effect of torsional coupling is absent for the unsprung mass system, the results also indicate that this effect for a sprung mass system is negligible for smooth pavement. Due to the eccentric movement of the vehicle, however, a torsional response is caused, leading to a difference in the response of the bridge deck (compare the responses in Table 2 with those given above for a vehicle eccentricity of 4·0 m). This torsional response is due to the pure torsional mode of response for the bridge as obtained from solving equation (2) independently.

In Table 3, the maximum deflections for the bridge are compared for different cases when the pavement irregularity is considered in the analysis. The pavement irregularity is considered by generating a typical rough bridge pavement with the following properties: $\bar{\varepsilon} = 4 \times 10^{-6}$ m$^2$/cm, $\beta = 1·94, \gamma = 0·01$ cycles/m, $f_a = 10·0$ cycles/m.

It is seen that the responses considerably differ between the unsprung and sprung mass models, and depend on the speed parameter and the frequency ratio. Thus, the pavement irregularity introduces a considerable difference in the bridge–vehicle interactive force between the sprung and unsprung mass models. For the sprung mass system, the interactive force is significantly dependent upon the values of the frequency ratio and the speed.

<table>
<thead>
<tr>
<th>Speed parameter</th>
<th>Unsprung mass system</th>
<th>Sprung mass system</th>
</tr>
</thead>
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<td></td>
<td>$f_r = 1·1267$</td>
<td>$f_r = 1·7814$</td>
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<td>0·003388</td>
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<td>0·003522</td>
<td>0·003508</td>
</tr>
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</table>

Table 3

**Comparison of maximum dynamic deflections (m) for a typical randomly rough bridge pavement (no eccentricity)**

<table>
<thead>
<tr>
<th>Speed parameter</th>
<th>Unsprung mass system</th>
<th>Sprung mass system</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_r = 1·1267$</td>
<td>$f_r = 1·7814$</td>
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<td>0·012758</td>
<td>0·004143</td>
</tr>
<tr>
<td>0·15</td>
<td>0·015054</td>
<td>0·004497</td>
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</table>
parameter. For the case of an unsprung mass system, the interactive force, and hence the response of the bridge, change considerably with the speed parameter. The reason for this can be explained with the help of equations (21) and (23). For an unsprung mass system, the relative displacement of the mass \( u \) is zero in equation (23), resulting in \( x = y + r \), or, \( \ddot{z} = \ddot{y} + \ddot{r} \); upon assuming that the effect of \( \ddot{y} \) is much smaller and can be neglected (as explained before), \( \ddot{z} \) becomes equal to \( \ddot{r} \). In other words, the interactive force on the bridge becomes equal to \( (W_r/g)\ddot{r} \). Since \( \ddot{r} \) depends upon the speed of the vehicle, the interactive force for an unsprung mass system changes with the speed parameter for an irregular bridge profile. Furthermore, comparisons between the responses shown in Tables 2 and 3 indicate that the pavement irregularity significantly influences the dynamic response of the bridge due to vehicle movement. Note that in Table 3 the responses for the sprung mass system are less than those for the unsprung mass system. This is due to the energy dissipation that takes place in the bilinear vehicle model (for the sprung mass system).

In Figure 4, the influence line for the dynamic deflection of the bridge is shown when the pavement irregularity is considered in the analysis. For eccentric movement of the vehicle (as in the case of smooth bridge pavement), the maximum deflections of the bridge at a distance of 5.0 m away from the centreline are obtained as 0.011884 and 0.004956 m for the unsprung and sprung mass models, respectively. The results for eccentric vehicle movement again show that the bilinear nature of the interactive force (for the sprung

### Table 4

*Effect of torsional coupling for rough bridge pavement \( k_f/k_r = 1.00 \)*

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Coupling condition</th>
<th>( y ) (m)</th>
<th>( \theta ) (rad)</th>
<th>( y ) (m)</th>
<th>( \theta ) (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>Uncoupled</td>
<td>0.005322</td>
<td>0.000202</td>
<td>0.007053</td>
<td>0.000288</td>
</tr>
<tr>
<td></td>
<td>Coupled</td>
<td>0.005306</td>
<td>0.000200</td>
<td>0.006892</td>
<td>0.000269</td>
</tr>
<tr>
<td>0.15</td>
<td>Uncoupled</td>
<td>0.005924</td>
<td>0.000221</td>
<td>0.007104</td>
<td>0.000270</td>
</tr>
<tr>
<td></td>
<td>Coupled</td>
<td>0.005901</td>
<td>0.000220</td>
<td>0.006932</td>
<td>0.000255</td>
</tr>
</tbody>
</table>
mass system) greatly reduces the maximum deflection of the bridge compared to that for the unsprung mass model of the vehicle.

In order to investigate the effect of torsional coupling on the bridge response, the maximum flexural and torsional displacements (at a distance of 5·0 m from the centreline of the bridge) were obtained with and without torsional coupling. For the latter, equations (1) and (2) are solved separately, and the responses due to both are added together for each time station. The results are shown in Table 4. It is seen that the effect of torsional coupling is not significant. This is due to the fact that the bridge is torsionally very stiff and, therefore, the torsional response is very small. Consequently, the effect of torsional coupling becomes insignificant.

In Figure 5 is shown the variation of the dynamic amplification factor (DAF) for the mid-point of the bridge (defined as the ratio of the maximum dynamic deflection to the maximum static deflection) with the frequency ratio $f_r$ for three different speed parameters. The $k_s/k_i$ ratio for the sprung mass system is assumed to be equal to 1. The DAF value is obtained after averaging the DAFs from the results of ten simulated random bridge profiles. The standard deviations (s.d.) for the peak response and the ratio of the standard deviation to average peak value of the response for the simulated results are shown in Table 5. It may be seen from the table that both the s.d. and the ratio of the s.d. to the average peak value for the response are extremely small. Therefore, the results from ten simulated random bridge profiles were adopted for the parametric study.

**Table 5**

<table>
<thead>
<tr>
<th>Speed parameter</th>
<th>$f_r = 1·1267$</th>
<th>$f_r = 1·7814$</th>
<th>$f_r = 2·2533$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.d. (m)</td>
<td>s.d. $/W_{pav}$</td>
<td>s.d. (m)</td>
<td>s.d. $/W_{pav}$</td>
</tr>
<tr>
<td>0·09</td>
<td>0·000061</td>
<td>0·014</td>
<td>0·000307</td>
</tr>
<tr>
<td>0·12</td>
<td>0·000091</td>
<td>0·022</td>
<td>0·000190</td>
</tr>
<tr>
<td>0·15</td>
<td>0·000301</td>
<td>0·070</td>
<td>0·000242</td>
</tr>
</tbody>
</table>

Note: $W_{pav}$ = average value of peak deflection of bridge.
It is seen from Figure 5 that the DAF generally increases with the increase of \( f_e \). The values of the DAF do not differ much for different speed parameters up to a certain value of \( f_e \), beyond which they differ significantly. Furthermore, for certain speed parameters, the DAF may drastically change with \( f_e \) for greater values of \( f_e \).

The variation of DAF with the speed parameter is plotted in Figure 6 for three different values of eccentricity; namely, 0, 2.0 m and 4.0 m. The deflection of the bridge is measured at a transverse distance of 5.0 m from the centreline of the bridge deck. The variations show that initially the DAF increases slowly with increase in the speed parameter \( a \). For higher values of \( a \), the DAF starts to increase sharply with increasing \( a \). The plots also show that the effect of eccentricity of the vehicle path on the DAF is not significant, as the three curves are closely spaced. This indicates that the torsional coupling effect for a torsionally stiff continuous bridge is not significant in comparison with its flexural vibration.

7. CONCLUSIONS

A continuum approach has been used to find the dynamic response of a multispans continuous bridge under a moving vehicle, modelled as a single unsprung or sprung mass. The response has been obtained in time domain, with due consideration of the non-linear effects of the bridge-vehicle interaction, of the torsion in the bridge and of simulated random irregularity of the bridge pavement. With the help of this method of analysis, a parametric study has been conducted to investigate the influence of some important parameters on the dynamic behaviour of the bridge. The following conclusions can be drawn from the results of the parametric study.

(i) For torsionally stiff (torsional frequencies much higher than the flexural frequencies) continuous bridges, the effect of torsional coupling due to the eccentricity in the vehicle path on the DAF is not significant.

(ii) For a smooth bridge pavement, the unsprung and sprung mass idealizations of the vehicle produce practically the same response for the bridge.

(iii) If the random irregularity of the bridge pavement is included in the analysis, the unsprung and sprung mass idealizations produce significantly different results. For both cases, the responses differ considerably from those for a smooth bridge pavement, and very much depend upon the values of the speed parameter and the frequency ratio.
(iv) For the sprung mass idealization of vehicles, the variation of the DAF with the frequency \( f \) follows different patterns for different speed parameters. For certain values of the speed parameter and the frequency ratio, the value of DAF may be excessively high.

(v) The value of the DAF generally increases with an increasing speed parameter. For high values of the speed parameter, the DAF shows a sharp increase.

REFERENCES


APPENDIX I

\[ I_1 = A_n(X_1 + X_2 - X_3 - X_4)/2, \quad I_2 = B_n(Y_1 - Y_2 - Y_3 + Y_4)/2, \]
\[ I_3 = C_n(Z_1 + Z_2 - Z_3 - Z_4)/2, \quad I_4 = D_n(Z_1 - Z_2 - Z_3 + Z_4)/2, \]
in which

\[
\begin{align*}
X_1 &= [\rho \sin \alpha t + \sigma_1 \cos \alpha t]/d_1, & X_2 &= [\rho \sin \alpha t - \sigma_2 \cos \alpha t]/d_2, \\
X_3 &= [\rho \sin \epsilon t + \sigma_3 \cos \epsilon t] \exp(-\rho t)/d_1, & X_4 &= [\rho \sin \epsilon t - \sigma_4 \cos \epsilon t] \exp(-\rho t)/d_2, \\
Y_1 &= [\rho \cos \alpha t + \sigma_1 \sin \alpha t]/d_1, & Y_2 &= [\rho \cos \alpha t + \sigma_2 \sin \alpha t]/d_2, \\
Y_3 &= [\rho \cos \epsilon t - \sigma_3 \sin \epsilon t] \exp(-\rho t)/d_1, & Y_4 &= [\rho \cos \epsilon t + \sigma_4 \sin \epsilon t] \exp(-\rho t)/d_2, \\
Z_1 &= \epsilon \exp(\epsilon t)/d_1, & Z_2 &= \epsilon \exp(-\epsilon t)/d_2, \\
Z_3 &= [\epsilon \cos \epsilon t + \sigma_3 \sin \epsilon t] \exp(-\rho t)/d_1, & Z_4 &= [\epsilon \cos \epsilon t - \sigma_4 \sin \epsilon t] \exp(-\rho t)/d_2,
\end{align*}
\]

where

\[
\begin{align*}
\sigma &= \beta \omega, & \epsilon &= \alpha \omega, & \rho &= \epsilon, \\
\sigma_1 &= \sigma + \epsilon, & \sigma_2 &= \sigma - \epsilon, & \sigma_3 &= \rho + \sigma, & \sigma_4 &= \rho - \sigma, \\
d_1 &= \rho^2 + \sigma_1^2, & d_2 &= \rho^2 + \sigma_2^2, & d_3 &= \epsilon^2 + \sigma_3^2, & d_4 &= \epsilon^2 + \sigma_4^2.
\end{align*}
\]
APPENDIX 2

\[ \{A_{mr}, B_{mr}, C_{mr}, D_{mr}\} \text{ integration constants used in the expression for mode shapes in transverse vibration} \]

\[ \{\bar{A}_{mr}, \bar{B}_{mr}\} \text{ integration constants used in the expression for mode shapes in torsional vibration} \]

\[ C_s \text{ distance of load from left to } r\text{th span} \]

\[ \{C\} \text{ vector of integration constants consisting of } A_{mr}, \text{ etc.} \]

\[ e \text{ eccentricity of vehicle path} \]

\[ e_{x} \text{ transverse distance from the centre-line of bridge where the deflection is sought} \]

\[ E \text{ modulus of elasticity of bridge material} \]

\[ \{F\} \text{ end force vector for } r\text{th span} \]

\[ f, f_1, f_0 \text{ spatial frequencies of pavement roughness} \]

\[ f_r \text{ frequency ratio} \]

\[ G \text{ shear modulus of elasticity} \]

\[ g \text{ gravitational acceleration} \]

\[ I, I_r \text{ moment of inertia of the bridge (beam) segment} \]

\[ I_{mr} \text{ mass moment of inertia of the bridge segment} \]

\[ J, J_r \text{ torsion constant of the bridge (beam) segment} \]

\[ [K] \text{ dynamic stiffness matrix for } r\text{th span} \]

\[ k_s, k \text{ tyre and suspension spring stiffness of the vehicle axle} \]

\[ k_{es} \text{ equivalent spring stiffness when both tyre and suspension spring are in action} \]

\[ l, l_r \text{ length of } r\text{th span} \]

\[ L \text{ total span of the bridge} \]

\[ M \text{ torsional moment acting on the bridge due to the eccentric vehicle position} \]

\[ m \text{ mass per unit length of the bridge} \]

\[ m_t(x, t) \text{ torsional load function} \]

\[ N \text{ number of spans in the bridge} \]

\[ n, r, s \text{ subscripts used to indicate } r\text{th vibrational mode, } r\text{th span and the span under the load, respectively} \]

\[ \rho(x, t) \text{ transverse load function} \]

\[ P \text{ magnitude of the moving concentrated load} \]

\[ P_m, P_s \text{ instantaneous and static force on the bridge} \]

\[ [R] \text{ integration constant matrix for } r\text{th span} \]

\[ r(t), r \text{ ordinate of random bridge pavement profile} \]

\[ S(f) \text{ power spectral density function} \]

\[ T_{s} \text{ displacement function in generalized co-ordinates} \]

\[ t, t_r \text{ time co-ordinate} \]

\[ u \text{ relative displacement of the vehicle centre of gravity} \]

\[ v \text{ velocity of the vehicle} \]

\[ W_s \text{ weight of the vehicle} \]

\[ W(x, t) \text{ dynamic deflection function} \]

\[ x, x_r \text{ distance along the span} \]

\[ y(x, t), y \text{ dynamic deflection of bridge under the vehicle} \]

\[ z \text{ vertical displacement of the vehicle centre of gravity} \]

\[ \delta \text{ pavement roughness coefficient} \]

\[ \alpha \text{ speed parameter} \]

\[ \beta \text{ a constant used in the expression of power spectral density function of surface roughness} \]

\[ \bar{\beta} \text{ a small value used in the iterative solution} \]

\[ \delta(t) \text{ Dirac delta function} \]

\[ \epsilon \text{ tolerance limit to check the convergence} \]

\[ \phi \text{ mode shape for transverse vibration} \]

\[ \Omega \text{ mode shape for torsional vibration} \]

\[ \omega, \omega_s, \omega_m \text{ natural frequency of bridge} \]

\[ \omega_l \text{ natural frequency of the idealized vehicle over its tyres} \]

\[ \zeta, \xi \text{ damping ratio} \]

\[ \mu \text{ vehicle constants which defines the limiting frictional force in the suspension system of the vehicle} \]

\[ \theta(x, t) \text{ torsional rotation function} \]