Theory and Methodology

A fuzzy set theoretic approach to qualitative analysis of causal loops in system dynamics

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Abstract: The paper presents an alternate method for the qualitative analysis of causal loops in system dynamics models. The focus of the work is to develop a methodology for qualitatively analysing causal loops to study the behavior of variables linguistically. It incorporates the uncertainties in a system dynamics model that could be attributed to the beliefs and perceptions of the user, and models them using fuzzy quantifiers and fuzzy relations.

Keywords: System Dynamics; Fuzzy sets; Qualitative analysis

1. Introduction

Most of the systems that are complex in nature exhibit dynamic behavior. Such systems have a structure associated with them. The structure could be transparent as in some of the transportation systems or could not be so as in socio-cultural systems. Also the systems could be of two types: open loop and closed loop systems. The open loop systems comprise of those in which an input gives rise to an output but in turn is not affected by it. They consist of a forward path only. On the other hand, closed loop systems have a feedback path of information, choice and action which connects the input to the output. Thus a closed chain of cause-and-effect or a feedback loop is generated. The dynamic behavior of an open loop system could be due to its response to external changes while the dynamics of a closed loop system arises because of its own attempts to control itself in the light of external changes. In other words the dynamic behavior could be attributed to the feedback loops [12,22].

Thus one of the ways in which the structure of the system could be unearthed is in the form of finding out all the cause-and-effect relationships. This is basically, the first step in system dynamics modelling which is followed by flow diagramming and simulation [12,22,24].

Over a period of time, system dynamics has gained a lot of popularity in the circle of policy makers and the academic community. A lot of developments have been made that aim at rendering help to
modellers in building computer based learning environments (or microworlds) for policy makers so that they can interplay and interact with their knowledge of business and social systems and to analyse policy and strategy changes. Though more and more developments would keep on being added, the most ambitious research path as conveyed by Morecroft leads into the realm of computer science and artificial intelligence, with a focus on conceptual/symbolic level use of the policymaker’s knowledge [19].

Similarly, Keloharju also suggests that system dynamics to be more competitive in the future would require inclusion of some features of data support systems and expert systems [9].

Thus the message that clearly emerges is that system dynamics in the future has to have an interface with expert systems in order to be more effective and useful. System dynamics as a tool can be used both for qualitative and quantitative analysis. However, presently qualitative analysis is done by interpreting causal loop diagrams. The approach proposed in this paper provides a systematic method of assessing the qualitative impact of relationships depicted in the causal loop diagram. This is a significant advantage over existing methods. As system dynamics is inherently qualitative the proposed approach is to enhance system dynamics with interfacing techniques, such that knowledge level qualitative use is enabled. The proposed analysis could also be applied to derive an expert system structure for the system dynamics model.

2. Theoretical background

2.1. An introduction to system dynamics

System dynamics has been defined as that branch of control theory which deals with socio-economic systems, and that branch of management science which deals with problems of controllability [2].

System dynamics rests on the premise that the dynamic behavior as exhibited by various systems is due to the presence of causal loops of interdependence of various variables in a system. The system dynamics approach is based on scanning within a system for the sources of its problem behavior. This internal viewpoint results in models of feedback systems that make external factors internalised. The behavior of a feedback system of interconnected feedback loops often confounds common intuition and analysis because of the complexity of the structure, and thus the behavior generated by them over time can usually be traced only by simulation. It differs from other mathematical modelling techniques in as much as it uses a systems concept which enquires into the components of system and their interrelationships with a view to answering a simple question: “What affects what?”

The procedure of system dynamics modelling could be broadly broken up into five major steps, i.e., identification of problem structure, influence diagramming or causal-loop diagramming, flow diagramming, simulation, and finally analysis and revision [3–5,10,17,21].

2.2. A brief overview of fuzzy set theory

Zadeh defines a fuzzy set as a ‘class’ with a continuum of grades of membership. A fuzzy subset $A$ of a universe of discourse $U$ is characterized by a membership function $\mu_A : U \rightarrow [0, 1]$ which associates with each element $x$ of $U$ a number $\mu_A(x)$ in the interval $[0, 1]$ which represents the grade of membership of $x$ in $A$ [26].

The support of $A$ is the set of points in $U$ at which $\mu_A(x)$ is positive.

A fuzzy singleton is a fuzzy set whose support is a single point in $U$. If $A$ is a fuzzy singleton whose support is the point $x$, then

$$A = \mu / x$$

where $\mu$ is the grade of membership of $x$ in $A$. 

\[1\]
A fuzzy set $A$ may be viewed as the union of its constituent singletons. Thus, it could be represented as

$$A = \int_{U} \mu_A(x)/x$$

(2)

where the integral sign stands for the union of the fuzzy singletons $\mu_A(x)/x$. If $A$ has a finite support $\{x_1, x_2, \ldots, x_n\}$, then (2) may be written as

$$A = \mu_1/x_1 + \mu_2/x_2 + \cdots + \mu_n/x_n$$

(3)

where '+' stands for union and not the arithmetic sum, or

$$A = \sum_{i=1}^{n} \mu_i/x_i$$

(4)

in which $\mu_i$, $i = 1, 2, \ldots, n$, is the grade of membership of $x_i$ in $A$.

There are a variety of fuzzy set operations, but the basic operations are complementation, union and intersection. The definitions of complementation, union, and intersection proposed by Zadeh and used since then are as follows [26]:

(a) The complement of $A$ is denoted by $\neg A$ and the membership function of $\neg A$ is given by

$$\mu_{\neg A}(x) = 1 - \mu_A(x) \quad \forall x \in X.$$ 

(5)

(b) The union of fuzzy sets $A$ and $B$ is denoted by $A \cup B$ and the membership function of $A \cup B$ is given by

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \quad \forall x \in X.$$ 

(6)

(c) The intersection of fuzzy sets $A$ and $B$ is denoted by $A \cap B$ and the membership function of $A \cap B$ is given by

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \quad \forall x \in X.$$ 

(7)

2.3. Expert systems concepts

Expert systems is one of the branches of artificial intelligence that aims at achieving expert level results in solving relatively difficult problems. These are computer programs that try to mimic the behavior of an expert. They have a built-in inferential capability by virtue of being programmed in such a manner and can even handle uncertain, unknown and conflicting data.

A large number of expert systems have been developed as rule-based production systems. Such a system comprises of: (i) a global data base that consists of a set of facts of assertions about the current problem; (ii) a rule base that consists of the experts' knowledge in the form of conditional type statements, each statement being comprised of an antecedent and a consequent portion, i.e., rules of the form 'if antecedent, then consequent'; and (iii) a rule interpreter that consists of the portion of the system that carries out the problem solving. The rule interpreter can be considered to have two components. The first component consists of the inference mechanism, which helps in determining when a particular rule is valid and what is the effect of applying this rule. The second component consists of some meta-rules that help determine in which order the rule base is to be searched for applicability of the rules [1,14–16,18,20,25,27].
3. A fuzzy model for causal loop analysis

3.1. Causal loop diagram

As already explained, influence or causal loop diagramming is the most important step in system dynamics modelling and considerably affects the construction and analysis of a model.

The causal loop diagram explains how the system works. This is accomplished through writing the names of the variables concerned and connecting them by an arrow or influence line (or link). The direction of the arrow shows the direction of causation. The signs at the heads (points) of the arrows show the sign of the effects. If the head variable changes in the same direction as the tail variable, a '+' (plus) sign is put on the link while a '−' (minus) sign is put if it changes in the opposite direction, e.g., Figure 1 depicts that increase in advertising increases sales while increase in price would reduce the sales.

Further, the variables may affect other variables and thus form loops or chains of interrelated variables. Figure 2 shows a small example of such a loop.

The variables quantity sold, sales revenue and advertising budget form a feedback loop which happens to be positive in this case as any increase in advertising budget increases the sales which in turn raises the sales revenue and consequently the increase in advertising budget is reinforced.

Similarly, there could also exist negative feedback loops.

3.2. Significance of fuzzy relations in qualitative analysis

The relationships as depicted by the causal loop diagram are often fuzzy in nature, as there are no crisp answers to questions like how much increase in advertising budget would raise the sales by a certain amount, say x% in the example taken earlier. Similarly, in practice it is very difficult to say or decide upon the amount of advertising budget as the proportion of sales revenue, for it is a function of multiple factors, like the current market situation, competition, market share, stage of product life cycle, policies of management, etc. Indeed, in real life the answers to such questions are likely to be of the form that a large increase in advertising budget may increase the sales by very large, moderate, small or very small amount.

Thus, the links could be viewed as fuzzy relations so that the model captures real life character.

3.3. Fuzzy relations

A fuzzy relation $R$ from a set $X$ to a set $Y$ is a fuzzy subset of the Cartesian product $X \times Y$. ($X \times Y$ is the collection of ordered pairs $(x, y), x \in X, y \in Y$). $R$ is characterised by a bivariate membership function $\mu_R(x, y)$ and is expressed as follows:

```
Advertising Budget → Sales ← Price
```

Figure 1. Cause and effect

```
Quantity Sold → {+} ← Sales Revenue

Advertising Budget
```

Figure 2. A causal loop
Let $X, Y \subseteq R$ be universal sets. Then

$$R = \{(x, y), \mu_R(x, y)\mid (x, y) \subseteq X \times Y\}$$

(8)

is called a fuzzy relation on $X \times Y$.

Let $X, Y \subseteq R$ and let $A = \{(x, \mu_A(x))\mid x \in X\}$, $B = \{(y, \mu_B(y))\mid y \in Y\}$ be two fuzzy sets. Then

$$R = \{(x, y), \mu_R(x, y)\mid (x, y) \in X \times Y\}$$

(9)

is a fuzzy relation on $A$ and $B$ iff

$$\mu_R(x, y) \leq \mu_A(x) \quad \forall (x, y) \in X \times Y \quad \text{and} \quad \mu_R(x, y) \leq \mu_B(y) \quad \forall (x, y) \in X \times Y.$$

3.3.1. Max–min composition

Let $R(x, y), (x, y) \subseteq X \times Y$ and $S(y, z) \subseteq Y \times Z$ be two fuzzy relations. Then the max–min composition of $R$ max–min $S$ denoted by $R \circ S$ is the fuzzy set

$$R \circ S = \left\{(x, z) \mid \max_y \left(\min\{\mu_R(x, y), \mu_S(y, z)\}\right) \right\} x \in X, y \in Y, z \in Z.$$  

(10)

3.3.2. Compositional rule of inference

Let $X$ and $Y$ be two universes of discourse with basic variables $x$ and $y$ respectively. Let $R(x), R(x, y)$ and $R(y)$ denote restriction on $x, (x, y)$ and $y$ respectively, with the understanding that $R(x), R(x, y)$ and $R(y)$ are fuzzy relations in $X, X \times Y$, and $Y$. Let $A$ and $F$ denote particular fuzzy subsets of $X$ and $X \times Y$. Then the compositional rule of inference asserts that the solution of the relational assignment equations [26]

$$R(x) = A$$

(11)

and

$$R(x, y) = F$$

(12)

is given by

$$R(y) = A \circ F$$

(13)

where $A \circ F$ is the max–min composition of $A$ and $F$, as explained by (10). In this sense it can be inferred that $R(y) = A \circ F$ from $R(x) = A$ and $R(x, y) = F$ [11,26].

3.3.3. Modus ponens and compositional rule of inference

The expression "If $A$ Then $B$ Else $C$" is a binary fuzzy relation in $X \times Y$ defined in [26]:

$$\text{If } A \text{ Then } B \text{ Else } C = A \times B + \neg A \times C,$$

(14)

i.e., if $A$, $B$ and $C$ are unary fuzzy relations in $X$, $Y$ and $Y$, then "If $A$ Then $B$ Else $C$" is a binary fuzzy relation in $X \times Y$ which is the union of the Cartesian product of $A$ and $B$ and the Cartesian product of the negation of $A$ and $C$.

Then "If $A$ Then $B$" may be viewed as a special case of "If $A$ Then $B$ Else $C$", which results when $C$ is taken to be the entire universe $Y$. Thus

$$\text{If } A \text{ Then } B = \text{If } A \text{ Then } B \text{ Else } Y$$

(15)

$$= A \times B + \neg A \times Y.$$  

(16)
In terms of the relation matrices of $A$, $B$ and $C$ (14) may be expressed as

$$\text{If } A \text{ Then } B \text{ Else } C = [A][B] + [-[A]][C].$$

(17)

More generally, if $A_1, \ldots, A_n$ are fuzzy disjoint subsets of $X$, and $B_1, \ldots, B_n$ are fuzzy subsets of $Y$, then

$$\text{If } A_1 \text{ Then } B_1 \text{ Else If } A_2 \text{ Then } B_2 \cdots \text{ Else If } A_n \text{ Then } B_n$$

$$= A_1 \times B_1 + A_2 \times B_2 + \cdots + A_n \times B_n.$$  

(18)

3.4. The procedure for qualitative analysis

Now looking at the causal loop diagram, one could easily infer that the causal relationship as depicted by an arrow could be regarded as an If-Then relationship, e.g.,

Advertising Budget $\rightarrow$ Sales

could also be seen as

IF Advertising Budget High THEN Sales is High.

In fact, we have established a production rule of If-Then type from the causal loop diagram. Once these rules are encoded using a suitable software, we arrive at the rule-base of the knowledge based system [15,25].

But since, as explained earlier, the relationships are fuzzy and also the If-Then (modus ponens) rule is a special case of the compositional rule of inference, the causal loop diagram could be suitably modelled using fuzzy relations. This would not only help in incorporating the beliefs and perceptions of the modeller in a scientific way, but also provide a methodology for qualitative analysis of system dynamics models [24]. Since most of the concepts in natural language are fuzzy in nature [26], a fuzzy set theoretic approach provides the best solution for such problems. Practising managers or experts can rarely tell with preciseness the exact relationship between variables in most of the cases and perhaps use linguistic variables [6,7], e.g., predicates like attractive offer, risky venture; quantifiers such as most, several, few; temporal quantifiers such as frequently, often, once in a while, almost always; and qualifiers such as almost true, quite right, likely, impossible, etc. A more detailed account on linguistic variables has been given in [26].

Thus, in many cases whenever they are asked to provide exact functional relationships, it seems as if managers have been put in some sort of artificial situation. Therefore, it would be much more suitable and appealing to managers and users if they could communicate with the expert (here the computer) in as much natural a way as possible. Use of linguistic variables frees the manager from such artificiality.

3.4.1. Model inputs and output

The major inputs required for the model are the problem situation defined, causal loop diagram, relevant data, definition of linguistic variables and the If-Then rules. Not all the inputs are easily obtainable as some of them are based on the beliefs and perceptions of the modeller. However, this should pose no problem as we know that fuzziness inherently deals with subjectivity and other decision making techniques, too, are not absolutely free from subjectivity. Moreover there could be inconsistencies and conflicts in the rules as they are not always specified by a single manager. However, there are different methods available for evolving a sort of consensus [8,23], which have not been taken into cognizance here as it is somewhat out of the scope of this particular paper.

The output provided would be the effect of a particular state of variable on the other variables which are influenced by it through a chain of causation. The output would be of the fuzzy subset form which could be interpreted back to a linguistic variable.

1 In the sense used in ALGOL, the right-hand side of (18) would be expressed as $A_1 \times B_1 + (\neg A_1 \cap A_2) \times B_2 + \cdots + (\neg A_1 \cap \cdots \cap \neg A_{n-1} \cap A_n) \times B_n$ when the $A_i$ and $B_i$, $i = 1, \ldots, n$, are nonfuzzy sets [26].
3.4.2. Steps

The proposed analysis involves the following steps:
(a) Description of the problem situation.
(b) Making a causal loop diagram for the given situation.
(c) Collection of relevant data for quantification of the various variables and parameters.
(d) Definition of the linguistic variables involved.
(e) Translation of the causal loop diagram into various rules of If-Then type.
(f) Generation of fuzzy relation matrices from the If-Then rules using modus ponens and compositional rule of inference.
(g) Analysis, i.e., using the fuzzy relations to study the change of one or more variables on other variables in the system.

The method proposed here is different from conventional methods as it starts digressing from step (c). In conventional methods the qualitative analysis is done directly by interpreting the causal loop diagram obtained from step (b).

4. Illustration

The procedure of qualitative analysis has been illustrated through a small example. The example has been kept fairly simple and thus comprises of very few variables, for the purpose is to illustrate the concept only rather than presenting a case analysis.

4.1. Problem situation

Here is a small hypothetical company which deals in a non-durable consumer good. It has been observed that the quantity sold of this particular good is affected by price in a negative manner and the market size in a positive manner. The market size in turn depends only upon the disposable income per capita. The quantity sold also depends upon the advertising budget which is decided upon the basis of the sales revenue for a time period. The sales revenue is decided by the price and the quantity sold.

4.2. Causal loop diagram

The problem situation could be represented in the following form of a causal loop diagram as shown in Figure 3.

Thus it is clear from the above causal loop diagram that the disposable income per capita and the price are the only exogenous variables in this particular case.

4.3. Data

The range of various variables as specified by the manager is given in Table 1. It may be noted that there is no one-to-one relationship in the values of different variables given in Table 1.

![Figure 3. Causal loop diagram for the problem situation](image-url)
Table 1
Range of variables

<table>
<thead>
<tr>
<th>Price (Rs.)</th>
<th>Quantity sold (items)</th>
<th>Sales revenue (Rs.)</th>
<th>Advertising budget (Rs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>50</td>
<td>5500</td>
<td>550</td>
</tr>
<tr>
<td>90</td>
<td>60</td>
<td>6000</td>
<td>600</td>
</tr>
<tr>
<td>100</td>
<td>75</td>
<td>6750</td>
<td>675</td>
</tr>
<tr>
<td>110</td>
<td>90</td>
<td>7200</td>
<td>720</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Disposable income per capita (Rs./month)</th>
<th>Market size (items)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>3000</td>
</tr>
<tr>
<td>1000</td>
<td>4000</td>
</tr>
<tr>
<td>1500</td>
<td>5000</td>
</tr>
<tr>
<td>2000</td>
<td>6000</td>
</tr>
</tbody>
</table>

4.4. Definition of linguistic variables

The manager has defined the various linguistic variables in the form of Very Low (VL), Low (L), Medium (M), High (H) and Very High (VH) for each of the variables which are as follows:

**Price:**

\[
VL = \frac{0.8}{80} + \frac{0.6}{90},
\]

\[
L = \frac{0.8}{80} + \frac{1}{90} + \frac{0.5}{100},
\]

\[
M = \frac{0.8}{90} + \frac{1}{100} + \frac{0.5}{110},
\]

\[
H = \frac{0.5}{100} + \frac{0.3}{110} + \frac{1}{110},
\]

\[
VH = \frac{0.3}{100} + \frac{0.7}{110}.
\]

**Quantity Sold:**

\[
VL = \frac{0.8}{50} + \frac{0.5}{60},
\]

\[
L = \frac{1}{50} + \frac{0.8}{60} + \frac{0.2}{75},
\]

\[
M = \frac{0.8}{50} + \frac{1}{60} + \frac{0.5}{75},
\]

\[
H = \frac{0.3}{60} + \frac{0.6}{75} + \frac{1}{90},
\]

\[
VH = \frac{0.4}{75} + \frac{0.7}{90}.
\]

**Sales Revenue:**

\[
VL = \frac{0.8}{5500} + \frac{0.3}{6000},
\]

\[
L = \frac{1}{5500} + \frac{0.5}{6000},
\]

\[
M = \frac{1}{6000} + \frac{0.7}{6750},
\]

\[
H = \frac{0.8}{6000} + \frac{1}{6750},
\]

\[
VH = \frac{0.6}{6750} + \frac{0.9}{7200}.
\]
Advertising Budget:

\[
VL = \frac{0.7}{500} + \frac{0.5}{600}, \quad \text{(22a)}
\]
\[
L = \frac{1}{500} + \frac{0.6}{600} + \frac{0.5}{750}, \quad \text{(22b)}
\]
\[
M = \frac{0.8}{500} + \frac{1}{600}, \quad \text{(22c)}
\]
\[
H = \frac{0.7}{600} + \frac{1}{750}, \quad \text{(22d)}
\]
\[
VH = \frac{0.5}{750} + \frac{0.7}{720}. \quad \text{(22e)}
\]

Disposable Income Per Capita:

\[
VL = \frac{0.6}{500} + \frac{0.4}{1000}, \quad \text{(23a)}
\]
\[
L = \frac{1}{500} + \frac{0.2}{1000}, \quad \text{(23b)}
\]
\[
M = \frac{1}{1000} + \frac{0.4}{1500}, \quad \text{(23c)}
\]
\[
H = \frac{0.6}{1000} + \frac{1}{1500} + \frac{0.6}{2000}, \quad \text{(23d)}
\]
\[
VH = \frac{0.6}{1500} + \frac{1}{2000}. \quad \text{(23e)}
\]

Market Size:

\[
VL = \frac{0.8}{3000} + \frac{0.4}{4000}, \quad \text{(24a)}
\]
\[
L = \frac{1}{3000} + \frac{0.8}{4000} + \frac{0.5}{5000}, \quad \text{(24b)}
\]
\[
M = \frac{1}{4000} + \frac{0.6}{5000}, \quad \text{(24c)}
\]
\[
H = \frac{0.7}{4000} + \frac{1}{5000} + \frac{0.6}{6000}, \quad \text{(24d)}
\]
\[
VH = \frac{0.6}{5000} + \frac{0.9}{6000}. \quad \text{(24e)}
\]

4.5. Specification of the rules

The manager has come out with the following set of rules that hold well for this particular case:

R1 – IF Price Very Low THEN Quantity Sold Very High ELSE IF Price Low THEN Quantity Sold High
ELSE IF Price Medium THEN Quantity Sold Medium ELSE IF Price High THEN Quantity Sold
Low ELSE IF Price Very High THEN Quantity Sold Very Low.

R2 – IF Quantity Sold Very High THEN Sales Revenue Very High ELSE IF Quantity Sold High THEN
Sales Revenue Very High ELSE IF Quantity Sold Medium THEN Sales Revenue Medium ELSE
IF Quantity Sold Low THEN Sales Revenue Low ELSE IF Quantity Sold Very Low THEN Sales
Revenue Very Low.

R3 – IF Sales Revenue Very High THEN Advertising Budget Very High ELSE IF Sales Revenue High
THEN Advertising Budget High ELSE IF Sales Revenue Medium THEN Advertising Budget
Medium ELSE IF Sales Revenue Low THEN Advertising Budget Low ELSE IF Sales Revenue
Very Low THEN Advertising Budget Very Low.

R4 – IF Advertising Budget Very High THEN Quantity Sold High ELSE Medium.

R5 – IF Disposable Income Very High THEN Market Size High ELSE Medium.

R6 – IF Market Size Very High THEN Quantity Sold Very High ELSE IF Market Size High THEN
Quantity Sold Very High ELSE IF Market Size Medium THEN Quantity Sold High ELSE IF
Market Size Low THEN Quantity Sold Medium ELSE If Market Size Very Low THEN Quantity
Sold Medium.
4.6. Generation of fuzzy relation matrices

Let us derive the fuzzy relation matrix for the first rule, i.e., R1. Using (18)–(20), rule R1 could be put in the following form:

\[ R1 = \left( \begin{array}{cccc} 0.4 & 0.7 & 0.5 & 0.8 \\ 0.6 & 0.9 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.3 & 0.5 \\ 0.8 & 0.5 & 0.7 & 0.0 \end{array} \right) \times \left( \begin{array}{cccc} 0.9 & 0.6 & 0.3 & 0.1 \\ 0.6 & 0.7 & 0.4 & 0.2 \\ 0.5 & 0.8 & 0.9 & 0.1 \\ 0.0 & 0.1 & 0.2 & 0.3 \end{array} \right) \times \left( \begin{array}{cccc} 0.8 & 0.9 & 0.5 & 0.7 \\ 0.7 & 0.6 & 0.1 & 0.0 \\ 0.6 & 0.8 & 0.2 & 0.3 \\ 0.1 & 0.0 & 0.1 & 0.2 \end{array} \right) \]

or equivalently (in a matrix form):

\[
\begin{align*}
\text{R1:} \\
80 & \begin{bmatrix} 0 & 0.3 & 0.6 & 0.8 \\ 0.8 & 0.8 & 0.6 & 0.1 \end{bmatrix} \\
90 & \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.8 & 0.8 & 0.6 & 0.1 \end{bmatrix} \\
100 & \begin{bmatrix} 0 & 0.8 & 0.5 & 0 \\ 1 & 0.8 & 0.5 & 0 \end{bmatrix}
\end{align*}
\]

Similarly, other relations too could be derived in the same manner. The following matrices are obtained when solved for the remaining rules listed above:

\[
\begin{align*}
\text{R2:} \\
5500 & \begin{bmatrix} 1 & 0.8 & 0.7 & 0 \\ 0.8 & 1 & 0.7 & 0.3 \\ 0.2 & 0.5 & 0.6 & 0.6 \\ 0 & 0 & 0.6 & 0.9 \end{bmatrix} \\
6000 & \begin{bmatrix} 0.8 & 1 & 0.8 & 0 \\ 0.7 & 0.7 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.7 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]
R4:

\[
\begin{array}{c|cccc}
\text{Quantity Sold} & 50 & 60 & 75 & 90 \\
\hline
550 & 0.8 & 1 & 0.5 & 0 \\
600 & 0.8 & 1 & 0.5 & 0 \\
675 & 0.5 & 0.5 & 0.5 & 0.5 \\
720 & 0.3 & 0.3 & 0.6 & 0.7 \\
\end{array}
= \begin{bmatrix}
\text{Advertising Budget} \\
\text{Disposable Income Per Capita}
\end{bmatrix}
\begin{bmatrix}
3000 & 4000 & 5000 & 6000 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0.6 & 0 \\
0 & 0.6 & 0.6 & 0.6 \\
0 & 0.7 & 1 & 0.6 \\
\end{bmatrix}
\tag{M4}
\]

R5:

\[
\begin{array}{c|cccc}
\text{Market Size} & 3000 & 4000 & 5000 & 6000 \\
\hline
500 & 0.8 & 1 & 0.5 & 0 \\
1000 & 0.8 & 0.8 & 0.6 & 1 \\
1500 & 0.5 & 0.5 & 0.6 & 0.7 \\
2000 & 0 & 0 & 0.4 & 0.7 \\
\end{array}
= \begin{bmatrix}
\text{Disposable Income Per Capita}
\end{bmatrix}
\begin{bmatrix}
3000 & 4000 & 5000 & 6000 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0.6 & 0 \\
0 & 0.6 & 0.6 & 0.6 \\
0 & 0.7 & 1 & 0.6 \\
\end{bmatrix}
\tag{M5}
\]

R6:

\[
\begin{array}{c|cccc}
\text{Quantity Sold} & 50 & 60 & 75 & 90 \\
\hline
3000 & 0.8 & 1 & 0.5 & 0 \\
4000 & 0.8 & 0.8 & 0.6 & 1 \\
5000 & 0.5 & 0.5 & 0.6 & 0.7 \\
6000 & 0 & 0 & 0.4 & 0.7 \\
\end{array}
= \begin{bmatrix}
\text{Market Size}
\end{bmatrix}
\begin{bmatrix}
3000 & 4000 & 5000 & 6000 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0.6 & 0 \\
0 & 0.6 & 0.6 & 0.6 \\
0 & 0.7 & 1 & 0.6 \\
\end{bmatrix}
\tag{M6}
\]

4.7. Analysis

Let us take an example where the manager wants to see the effect of very low price and low disposable income per capita. As already defined by (19a) and (23b),

\[
\text{Very Low Price} = \frac{0.8}{60} + \frac{0.6}{90},
\]

\[
\text{Low Disposable Income per Capita} = \frac{1}{300} + \frac{0.8}{1000}.
\]

Now invoking the compositional rule of inference as given by (13), we successively obtain the effect of low disposable income per capita on the market size and then obtain the corresponding quantity sold by this firm pertaining to this market size.

\[
\begin{bmatrix}
\text{Disposable Income Per Capita}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 0.6 & 0 \\
0 & 0.6 & 0.6 & 0.6 \\
0 & 0.7 & 1 & 0.6 \\
\end{bmatrix}
\begin{bmatrix}
\text{Market Size}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 0.6 & 0 \\
0 & 0.6 & 0.6 & 0.6 \\
0 & 0.7 & 1 & 0.6 \\
\end{bmatrix}
\tag{26}
\]

\[
\begin{bmatrix}
\text{Market Size}
\end{bmatrix}
\begin{bmatrix}
0.8 & 1 & 0.5 & 0 \\
0.8 & 0.8 & 0.6 & 1 \\
0.5 & 0.5 & 0.6 & 0.7 \\
0 & 0 & 0.4 & 0.7 \\
\end{bmatrix}
\begin{bmatrix}
\text{Quantity Sold}
\end{bmatrix}
= \begin{bmatrix}
0.8 & 0.8 & 0.6 & 1 \\
\end{bmatrix}
\tag{27}
\]

But, the quantity sold is also determined on the basis of the price and thus the effect of very low price
on the quantity sold has also to be seen. The corresponding quantity sold is obtained by using (19a) and fuzzy relation matrix (M1):

\[
\begin{bmatrix}
0 & 0.3 & 0.6 & 0.8 \\
0.8 & 0.8 & 0.6 & 1 \\
0.8 & 1 & 0.5 & 0.5 \\
1 & 0.8 & 0.5 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0.6 \\
0.6 \\
0.6 \\
0.8 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0.6 \\
0.6 \\
0.6 \\
0.8 \\
\end{bmatrix}.
\]

(28)

Since Quantity Sold is determined by two exogenous variables independently, a constraining effect comes into the picture. In this particular case an intersection operator is required which gives the final constrained value of the quantity sold that is to be fed into the causal chain to find out its effect on other variables. The constraining is done as follows (using (7)):

\[
\text{Quantity Sold (Constrained)} = \text{Quantity Sold (Determined by Price)} \cap \text{Quantity Sold (Determined by Disposable Income Per Capita)}
\]

\[
\begin{bmatrix}
0.8 \\
0.8 \\
0.6 \\
1 \\
\end{bmatrix} \cap \begin{bmatrix}
0.6 \\
0.6 \\
0.6 \\
0.8 \\
\end{bmatrix} = \begin{bmatrix}
0.6 \\
0.6 \\
0.6 \\
0.8 \\
\end{bmatrix}
\]

(29)

Now feeding the constrained value of Quantity Sold in the causal chain the following results are obtained:

\[
\begin{bmatrix}
0.6 & 0.6 & 0.6 & 0.8 \\
\end{bmatrix} \circ \begin{bmatrix}
1 & 0.8 & 0.7 & 0 \\
0.8 & 1 & 0.7 & 0.3 \\
0.2 & 0.5 & 0.6 & 0.6 \\
0 & 0 & 0.6 & 0.9 \\
\end{bmatrix} = \begin{bmatrix}
0.6 & 0.6 & 0.5 & 0.8 \\
\end{bmatrix},
\]

(30)

\[
\begin{bmatrix}
0.6 & 0.6 & 0.6 & 0.8 \\
\end{bmatrix} \circ \begin{bmatrix}
1 & 0.8 & 0.3 & 0 \\
0.8 & 1 & 0.8 & 0 \\
0.7 & 0.7 & 1 & 0.6 \\
0 & 0 & 0.5 & 0.7 \\
\end{bmatrix} = \begin{bmatrix}
0.6 & 0.6 & 0.6 & 0.7 \\
\end{bmatrix},
\]

(31)

\[
\begin{bmatrix}
0.6 & 0.6 & 0.6 & 0.7 \\
\end{bmatrix} \circ \begin{bmatrix}
0.8 & 1 & 0.5 & 0 \\
0.8 & 1 & 0.5 & 0 \\
0.5 & 0.5 & 0.5 & 0 \\
0.3 & 0.3 & 0.6 & 0.7 \\
\end{bmatrix} = \begin{bmatrix}
0.6 & 0.6 & 0.6 & 0.7 \\
\end{bmatrix}.
\]

(32)

The value of Quantity Sold obtained is in the fuzzy subset form which could also be denoted as \(\frac{0.6}{50} + \frac{0.6}{75} + \frac{0.7}{90}\).

This could roughly be interpreted as "somewhat very high". All the other values of the intermediate variables in the form of fuzzy subsets could be interpreted in a similar manner.

Here, after completing this chain, we have once again reached the quantity sold which would be its value for the next time period. In the next time period, if the exogenous variables, i.e., Disposable Income Per Capita or Price change their values, then they would again constrain this value of Quantity Sold and a fresh value of Quantity Sold would be obtained which may again be re-fed in the chain so that the values for each variable in each time period can be calculated recursively as is done in system dynamics; e.g., here in this case if Price remains constant but Disposable Income Per Capita becomes high in the next time period, then the value of Quantity Sold in the second iteration turns out to be (in the fuzzy subset form) \(\frac{0.6}{50} + \frac{0.6}{75} + \frac{0.6}{90}\).

Similar iterations could be made for other time periods if desired.
5. Concluding remarks

An attempt has been made to develop a methodology for qualitative analysis of system dynamics models. Besides this, the perceptions and beliefs of the modeller have also been incorporated in the system dynamics framework in an objective way using concepts of fuzzy sets. Such a method could have enormous applications in system dynamics modelling especially where human beings come into picture, e.g., in management, psychology, economics, medical science, etc.

Some of the advantages that accrue from using this methodology are:

(a) It provides a systematic procedure for qualitative analysis of system dynamics models.

(b) Subjective beliefs and perceptions could be incorporated easily in an objective, scientific and rational manner using the concepts of fuzzy sets.

(c) It offers a sort of policy laboratory in which managers can interplay and interact with the model and experiment with various policies, so as to evolve a strategy and policies related to it.

(d) In some of the cases the policy experiments done at the qualitative stage may provide great insight into the system structure and behavior and facilitate changes at this stage only, thereby avoiding the unnecessary efforts and computations involved in simulation of the whole model.

(e) The dynamics also comes into picture at the qualitative analysis stage only, which is not so clearly depicted in a causal loop diagram.

(f) Once the causal loop diagram is translated into If-Then type rules, it is nothing but the rule base for an expert system and thus generates an expert system structure for the system dynamics framework.

(g) If-Then-Else kind of rules are also permissible in this framework, which cannot be incorporated into a causal loop diagram. Such kind of rules are taken into account only in the flow diagramming stage with the help of various functions like clip function, etc., but here they could be used right at the qualitative analysis stage only.

(h) Through the use of linguistic variables, it frees users from the artificiality involved in precise quantification of variables which is not always possible, and thus it would have a great appeal for users or managers.

(i) The method proposed here could be easily computerised for convenient and efficient handling of very large size models.

With a lot of scope being there to augment and refine the proposed methodology, we would like to make the comment that this method is by no means perfect and complete and at best could be taken as a starting point.

References


