A MUSIC-like method for estimating quadratic phase coupling

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Abstract

In this paper, a two-dimensional version of the well-known MUSIC algorithm for estimating the quadratically coupled frequency pairs (QC pairs) in a noise-corrupted complex harmonic process is proposed. It is shown that the algorithm can also, with minor modifications, be used for estimating the bispectrum of a general third-order stationary harmonic process. The algorithm involves arranging the complex third-order cumulants of the noisy harmonic process in the form of a matrix having a normal structure, i.e., a matrix with an orthonormal eigenbasis. It is shown that a necessary and sufficient condition for an ordered pair \((\theta_1, \theta_2)\) in the two-dimensional frequency plane to be a QC pair is that the Kronecker product between steering vectors associated with the two frequencies \(\theta_1\) and \(\theta_2\) lies in the signal subspace of this matrix. By exploiting this result and the orthogonality between the signal and noise subspaces of the above matrix, a symmetric search function of two frequency variables (termed as the MUSIC pseudo-bispectrum estimator) is constructed using the signal eigenvectors. This function is shown to peak precisely at the QC pair locations in the two-dimensional frequency plane. Simulation results are presented, and are shown to testify to the high-resolution performance of this estimator.

Zusammenfassung

Eine zweidimensionale Version des bekannten MUSIC-Algorithmus' zur Schätzung quadratisch gekoppelter Frequenzen (QC-Paare) in einem verrauschten komplexen harmonischen Prozess wird vorgeschlagen. Es zeigt sich, dass man das Verfahren mit geringfügigen Modifikationen auch zur Schätzung des Bispektrums eines allgemeinen stationären harmonischen Prozesses dritter Ordnung verwenden kann. Der Algorithmus beinhaltet die Anordnung der komplexen Kumulantens dritter Ordnung des verrauschten harmonischen Prozesses in Matrixform mit Normalstruktur, d.h. in einer Matrix mit orthonormaler Eigenbasis. Es wird gezeigt, dass eine notwendige und hinreichende Bedingung dafür, dass ein geordnetes Paar \((\theta_1, \theta_2)\) in der zweidimensionalen Frequenzebene ein QC-Paar darstellt, darin besteht, dass das Kroneckerprodukt der Steuervektoren zu den beiden Frequenzen \(\theta_1\) und \(\theta_2\) im Signal-Unterraum der Matrix liegt. Indem dieses Resultat und die Orthogonalität zwischen Signal- und Rauschunterraum der Matrix ausgenutzt werden, lässt sich eine symmetrische Suchfunktion der beiden Frequenzvariablen (als MUSIC-Pseudobispektrums-Schätzer bezeichnet) unter Verwendung der Signal-Eigenvektoren konstruieren. Es wird gezeigt, dass diese Funktion ihre Maxima exakt in den Punkten des QC-Paares in der komplexen Frequenzebene besitzt. Simulationsergebnisse werden vorgestellt. Sie belegen die Leistungsfähigkeit und die hohe Auflösung des Schätzers.

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Résumé

Nous proposons dans cet article une version bidimensionnelle de l'algorithme bien connu MUSIC pour l'estimation de paires de fréquence en couplage quadratique (paires QC) dans un processus harmonique complexe bruité. Nous montrons que l'algorithme peut également, au prix de modifications mineures, être utilisé pour estimer le bispectre d'un processus harmonique stationnaire du troisième ordre général. Cet algorithme requiert l'arrangement des cumulants complexes d'ordre trois du processus harmonique bruité sous la forme d'une matrice ayant une structure normale, à savoir une matrice ayant des vecteurs propres orthogonaux. Nous montrons qu'une condition nécessaire et suffisante pour qu'une paire ordonnée \((\theta_1, \theta_2)\) dans le plan fréquentiel bidimensionnel soit une paire QC est que le produit de Kronecker entre les vecteurs associés aux deux fréquences \(\theta_1\) et \(\theta_2\) se trouve dans le sous-espace de signal de la matrice. Nous appuyant sur ce résultat et sur l'orthogonalité des sous-espaces de signal et de bruit de la matrice mentionnée ci-dessus, nous construisons une fonction de recherche symétrique des deux variables de fréquence (appelée estimateur MUSIC du pseudo-bispectre) à l'aide des vecteurs propres correspondant au signal. Nous montrons que cette fonction possède un pic situé précisément sur les paires QC dans le plan fréquence bidimensionnel. Nous présentons des résultats de simulation qui mettent en évidence les capacités de haute résolution de cet estimateur.

Key words: Non-Gaussian; Third-order cumulants; Bispectrum; Quadratic phase coupling; Skewness; High resolution; MUSIC; Signal subspace; Noise subspace; Kronecker product; Pseudo-bispectrum

1. Introduction

During the past decade, cumulants and polyspectra have proved to be powerful statistical tools in non-Gaussian signal processing. They have been used for extracting information due to deviations from Gaussianity, identifying and characterizing nonlinear systems and estimating the phase of non-Gaussian, parametrically generated signals. As such, they have found wide applicability in diverse fields like sonar, radar, plasma physics, biomedicine, seismic data processing, array processing, oceanography, economic time series and sunspot data analysis. Some useful references are [1, 6–8, 10–12, 14, 20]. Some interesting references may also be found in [14]. A detailed survey of the existing literature on non-Gaussian signal processing using cumulants and polyspectra is available in [12, 14], and the references therein.

A major motivation behind the use of higher-order spectra (denoted HOS for convenience) in signal processing may be attributed to the fact that the second-order statistics are 'phase-blind', i.e., they contain no information about the phase of a signal being analyzed [14]. The power spectrum of a wide-sense stationary process gives only the distribution of power among the various harmonic components present in it. It does not tell us anything regarding the phase relations that may exist among these components. HOS, on the other hand, are 'phase-sensitive', and hence particularly useful for applications involving non-Gaussian signals and nonlinear and nonminimum phase systems [14, 19].

An important application of HOS concerns the response of a nonlinear system to a harmonic process, i.e., a process consisting of a finite number of sinusoids with random amplitudes and phases. The output is once again a superposition of sinusoids, the frequencies and phases of which are harmonically related to those of the input. Phase relations are thus produced among the harmonic components of the output due to nonlinear interactions between those of the input. The HOS, which are affected by phase relations, can thus be used to identify the nonlinearity. In this context, higher-order spectral analysis has been an important signal processing tool in diverse fields. Flow of energy in underwater, atmospheric, geophysical, optical or even biological media takes place in the form of propagating waves and one of the goals of signal processing is to detect and characterize these waves from sensor data measurements. Distortion of these waves due to nonlinearities in the propagating medium leads to nonlinear coupling in these waves. Power spectrum (or autocorrelation) based signal processing cannot distinguish nonlinearly coupled waves from spontaneously excited, independent
waves at the same frequencies. This distinction is achieved by HOS [10, 11, 13, 14]. In the field of biomedical signal processing, it is worth mentioning that polyspectral analysis has recently found an interesting application in the analysis of EEG data. Bispectral analysis of clinical EEG data has shown in some cases [8] to indicate a strong phase coupling between the so-called rhythms occurring in the brain. This information has been useful in obtaining more accurate parametric models of EEG signals than would have been possible by using just the second-order statistics. Parametric models of EEG signals based on HOS are expected to tell us more about the mechanism behind their generation [8].

In this paper, we focus our attention on the use of third-order cumulants (or moments) for the high-resolution estimation of quadratically phase-coupled sinusoids, which occur, for example, in second-degree nonlinear systems. The sinusoids \( x_1(n) = A_1 \exp{j(\omega_1 n + \phi_1)} \) and \( x_2(n) = A_2 \exp{j(\omega_2 n + \phi_2)} \), where \( \phi_1 \) and \( \phi_2 \) are independent random variables uniformly distributed in \([0, 2\pi)\), are said to be quadratically coupled in the signal \( x(n) = x_1(n) + x_2(n) + x_3(n) \), when \( x_3(n) = A_3 \exp{j(\omega_1 + \omega_2)n + \phi_1 + \phi_2} \), and \( A_1, A_2 \) and \( A_3 \) are non-zero, nonrandom real numbers. In such a case, the pair \((\omega_1, \omega_2)\) is referred to as a quadratically coupled (QC) pair. The power spectrum of \( x(n) \) merely shows impulses of strengths \( |A_1|^2, |A_2|^2 \) and \( |A_3|^2 \) at the frequencies \( \omega_1, \omega_2 \) and \( \omega_1 + \omega_2 \) but does not yield any information at all about the specific phase relation that exists between the harmonic components of \( x_1(n), x_2(n) \) and \( x_3(n) \). This phase information is contained in the bispectrum or, equivalently, in the third-order statistics of \( x(n) \).

The problem considered in the paper can be briefly stated as follows: Consider a signal consisting of a superposition of several complex sinusoids, some of which may be quadratically coupled on a pairwise basis in the sense that, to each pair of sinusoids, there exists another sinusoid whose frequency is the sum of their frequencies and whose phase is the sum of their phases. Devise an algorithm to extract all the QC pairs in this signal.

Several techniques for estimating quadratic phase coupling based on nonparametric and parametric bispectrum estimation exist in the literature [14, 18, 19]. The parametric time series modeling techniques are usually based on some a priori information regarding the physical generation of the process and, as such, do not suffer from the uncertainty principle of the Fourier transform, which limits the resolution of nonparametric estimators. However, while dealing with sinusoidal signals, time series modeling based methods do not appear to be appropriate since they result in unstable models, i.e., models with poles on the unit circle. On the other hand, eigenvector based methods provide a more natural setting while dealing with sinusoids, as they are based on a priori knowledge regarding the exact structure of the statistics of sinusoids in noise.

Recently, some attention has been focussed on developing eigenstructure algorithms based on higher-order statistics for sinusoidal frequency estimation and related problems [3–5, 15, 17, 24, 25]. These algorithms take care of the signal structure, while simultaneously being insensitive to additive Gaussian noise, colored or white. A MUSIC-like method which uses fourth-order cumulants is developed in [17] for direction of arrival estimation. In [5], the third-order statistics are used to construct beamformers for the bearing estimation problem. Particular mention should be made of [4], where elegant approaches for handling fourth-order cumulant statistics using index-free tensor products are discussed. It develops new algebraic techniques for blind identification of the direction vectors (i.e., without a priori knowledge of the array manifold) in the bearing estimation problem. The ideas in [4] are based on exploiting symmetry properties of the fourth-order cumulant tensor to derive its rank and eigenstructure.

This paper attempts to extend these contributions by developing generalized eigensubspace estimators, and in particular a MUSIC-like estimator based on third-order statistics, for estimating the quadratically phase-coupled frequency pairs in a sinusoidal process.

The MUSIC-like algorithm proposed in this paper is an algorithm based on the eigenstructure of a 'third-order cumulant matrix'. In the ideal given statistics situation, this eigenstructure can be exactly determined, and in this case it provides perfect resolution of the QC pairs. The principal idea in the
paper involves constructing a matrix of complex third-order cumulants that has a complete orthonormal eigenstructure (i.e., the matrix has a normal structure). Based on its eigenvalues, the column space of this matrix is separated into two mutually orthogonal subspaces, the signal subspace and the noise subspace. The signal subspace eigenvectors are then used to construct the MUSIC-like pseudo-bispectrum estimator. For this purpose, we make extensive use of the properties of the Kronecker tensor product.

2. Problem formulation

In what follows, we first formulate the general problem of estimating the bispectral parameters of a harmonic signal, and later specialize it to the situation in which we are primarily interested in estimating the phase-coupled terms. As a consequence, we shall show that although the former problem is more general than the latter, mathematically both can be viewed as equivalent problems in the sense that an algorithm that has been developed for solving any one of them can also be used for solving the other. The basic observation involved in demonstrating this equivalence is that the third-order moments of all third-order stationary harmonic signals have a similar mathematical structure. It is in view of this equivalence that we develop our algorithms with the quadratic coupling model in mind.

2.1. The general bispectral estimation problem

Let \( s(n) \) be a signal consisting of \( p \) sinusoids with random complex amplitudes \( \{ B_i \} \), and let \( x(n) \) be the observed signal consisting of \( s(n) \) in the presence of additive measurement noise \( w(n) \):

\[
\begin{align*}
  s(n) &= \sum_{i=1}^{p} B_i \exp\{j\omega_i n\}, \\
  x(n) &= s(n) + w(n).
\end{align*}
\]

The following assumptions regarding this model are made:

(a1) The complex amplitudes \( \{ B_i \} \) are random variables with a joint probability distribution such that \( s(n) \) has third-order stationary complex cumulants, i.e., Cum\{\( s^*(n), s(n + \tau), s(n + \rho) \)\} is only a function of \( \tau \) and \( \rho \). Necessary and sufficient conditions on \( \{ B_i \} \) in terms of the frequencies \( \{ \omega_i \} \), for which \( s(n) \) satisfies this hypothesis, have been derived and discussed in [5].

(a2) The frequencies \( \{ \omega_1, \ldots, \omega_p \} \) are distinct and nonrandom with values in \([0, 2\pi]\).

(a3) The additive noise \( w(n) \) is a zero-mean, third-order stationary and third-order white process independent of the signal process \( s(n) \). This implies that Cum\{\( w^*(n), w(n + \tau), w(n + \rho) \)\} = \( \gamma_w \delta(\tau) \delta(\rho) \), where \( \gamma_w \) is a fixed constant, called the noise skewness.

With these assumptions, it is readily seen that the third-order complex cumulants of \( x(n) \) have the form

\[
C_x(\tau, \rho) = C_s(\tau, \rho) + \gamma_w \delta(\tau) \delta(\rho),
\]

where

\[
C_s(\tau, \rho) = \text{Cum}\{x^*(n), x(n + \tau), x(n + \rho)\}
\]

and

\[
C_s(\tau, \rho) = \text{Cum}\{s^*(n), s(n + \tau), s(n + \rho)\}
\]

\[
= \sum_{i=1}^{q} \alpha_i \exp\{j(\theta_{1,i} \tau + \theta_{2,i} \rho)\},
\]

where \( \theta_{1,i}, \theta_{2,i} \in \{ \omega_j \mid 1 \leq j \leq p \} \), \( \theta_{1,i}, \theta_{2,i} \), \( 1 \leq i \leq q \), are distinct ordered pairs in \([0, 2\pi) \times [0, 2\pi]\), and each \( \alpha_i \) is a nonzero complex number.

The objective of the general problem is to obtain the bispectral parameters of \( s(n) \), namely, \( (\theta_{1,i}, \theta_{2,i}) \) and \( \alpha_i, i = 1, \ldots, q \), from a finite subset of the cumulant lags \( \{ C_x(\tau, \rho) \} \).

2.2. The problem of quadratic coupling estimation

The signal \( s(n) \) is next assumed to consist of a superposition of \( p \) sinusoids with nonrandom amplitudes, \( r \) pairs of which are quadratically coupled and the remaining \( p - 3r \) sinusoids are uncoupled. The observed noise-corrupted signal \( x(n) \) can thus be written as

\[
x(n) = s(n) + w(n),
\]

(1a)
where
\[ s(n) = \sum_{i=1}^{r} [s_{1,i}(n) + s_{2,i}(n) + s_{3,i}(n)] + \sum_{i=3r+1}^{p} s_i(n), \]  
with the coupled terms given by
\[ s_{j,i}(n) = A_{j,i} \exp\{j(\omega_j n + \phi_{j,i})\}, \]
\[ 1 \leq j \leq 3, \quad 1 \leq i \leq r, \quad \omega_{3,i} = \omega_{1,i} + \omega_{2,i}, \quad \phi_{3,i} = \phi_{1,i} + \phi_{2,i}, \quad 1 \leq i \leq r, \]  
and the uncoupled terms by
\[ s_i(n) = A_i \exp\{j(\omega_i n + \phi_i)\}, \quad 3r + 1 \leq i \leq p. \]  

The following assumptions are involved in this model:

(A1) The signal amplitudes \( \{A_{j,i}: 1 \leq j \leq 3, 1 \leq i \leq r\} \) and \( \{A_i: 3r + 1 \leq i \leq p\} \) are real, non-random and non-zero.

(A2) \( \{\phi_{j,i}: j = 1, 2, 1 \leq i \leq r\} \cup \{\phi_i: 3r + 1 \leq i \leq p\} \) is a collection of i.i.d. (identical, independent) random variables, each uniformly distributed in the interval \([0, 2\pi)\).

(A3) The frequencies in the signal \( s(n) \), \( \{\omega_{j,i}: 1 \leq j \leq 3, 1 \leq i \leq r\} \) and \( \{\omega_i: 3r + 1 \leq i \leq p\} \), all lie in the interval \([0, 2\pi)\) and the unordered pairs \( \{(\omega_{1,i}, \omega_{2,i}): i = 1, \ldots, r\} \) are distinct. (By the latter we mean that if, for some \( i \neq j \), \( \omega_{1,i} = \omega_{1,j} \), then \( \omega_{2,i} \neq \omega_{2,j} \) and, similarly, if \( \omega_{1,i} = \omega_{2,j} \), then \( \omega_{2,i} \neq \omega_{1,j} \).) In other words, corresponding to any pair of frequencies, we can have at most one triplet of sinusoids \( \{s_{1,i}(n), s_{2,i}(n), s_{3,i}(n)\} \) such that these two frequencies are quadratically coupled in \( s_{1,i}(n) + s_{2,i}(n) + s_{3,i}(n) \). That this assumption is in no way overly restrictive in as far as we deal with the third-order moments and the bispectrum, shall soon be demonstrated by showing the equivalence of this model to the general one described in Section 2.1.

The set of unordered pairs \( \{(\omega_{1,i}, \omega_{2,i}): 1 \leq i \leq r\} \) is called the set of quadratically coupled frequency pairs (QC pairs) and each member of this set, which is an unordered doublet, is called a QC pair. The triplet \( \{s_{1,i}(n), s_{2,i}(n), s_{3,i}(n)\} \) shall be termed as the quadratically coupled triplet associated with the QC pair \( (\omega_{1,i}, \omega_{2,i}) \). The signal \( s_{0,i}(n) = s_{1,i}(n) + s_{2,i}(n) + s_{3,i}(n) \) shall be called the signal associated with the QC pair \( (\omega_{1,i}, \omega_{2,i}) \). We sometimes say that the sinusoids \( s_{1,i}(n) \) and \( s_{2,i}(n) \) of frequencies \( \omega_{1,i} \) and \( \omega_{2,i} \), respectively, are quadratically coupled in the signal \( s(n) \) to give the sinusoid \( s_{3,i}(n) \) of frequency \( \omega_{1,i} + \omega_{2,i} \).

(A4) The noise process \( w(n) \) satisfies assumption (a3) of Section 2.1, namely, it is a zero-mean, third-order stationary and third-order white process, independent of the signal process \( s(n) \), and has a skewness \( \gamma_w \).

(A5) \( |A_{1,i}| A_{2,i} A_{3,i}| \gg |\gamma_w|, \quad 1 \leq i \leq r. \) Noting that \( E\{s_{0,i}(n)s_{0,i}(n + \tau)s_{0,i}(n + \rho)\} = A_{1,i} A_{2,i} A_{3,i} \times \exp\{j(\omega_{1,i} \tau + \omega_{2,i} \rho)\} \), this assumption can be equivalently stated as: The magnitudes of the skewness of the signals \( s_{0,i}(n) \) associated with each QC pair are much larger than the magnitude of the noise skewness. If \( w(n) \) is Gaussian, then \( \gamma_w = 0 \) and this condition is clearly satisfied. More generally, if the noise has a symmetric distribution about the origin or, equivalently, if the probability density of \( w(n) \) is an even function, then \( \gamma_w = 0 \) and the hypothesis is satisfied. It is simpler to assume that \( w(n) \) is Gaussian and work with \( \gamma_w = 0 \) as most authors do. However, for the sake of completeness, we stick to the non-Gaussian noise case, even though this assumption is still somewhat restrictive.

**Remark R1.** It is convenient to divide the QC pairs into two sets: viz., those which lie on the line \( \Omega_1 = 02 \) and those which do not. Notationally, we may say that \( \omega_{1,i} = \omega_{2,i} \) for \( i = 1, \ldots, s \) and \( \omega_{1,i} \neq \omega_{2,i} \) for \( i = s + 1, \ldots, r \).

Let
\[ C_s(\tau, \rho) = \text{Cum}\{x^*(n), x(n + \tau), x(n + \rho)\} = E(x^*(n)x(n + \tau)x(n + \rho)) \]  
and
\[ C_d(\tau, \rho) = \text{Cum}\{s^*(n), s(n + \tau), s(n + \rho)\} = E(s^*(n)s(n + \tau)s(n + \rho)). \]  
(That the third-order cumulants are the same as the corresponding moments is a consequence of the fact that \( s(n) \) and \( x(n) \) are zero-mean processes.)

From (1) and assumptions (A1), (A2) and (A4), it can readily be seen that
\[ C_s(\tau, \rho) = C_d(\tau, \rho) + \gamma_w \delta(\tau) \delta(\rho) \]  
(4a)
and

\[ C_s(\tau, \rho) = \sum_{i=1}^{r} \lambda_i \left[ \exp \{ j(\omega_{1,i} \tau + \omega_{2,i} \rho) \} ight. \\
+ \exp \left\{ j(\omega_{1,i} \rho + \omega_{2,i} \tau) \right\}, \tag{4b} \]

where \( \lambda_i \), which shall be referred to as the skewness associated with the QC pairs \((\omega_{1,i}, \omega_{2,i})\), equals \( A_{1,i} A_{2,i} A_{3,i} \).

It may be noted that the \( p - 3r \) uncoupled sinusoids \( s_i(n) \), \( 3r + 1 \leq i \leq p \), do not contribute to the complex third-order cumulants \( C_s(\tau, \rho) \) and hence also to \( C_x(\tau, \rho) \).

We can write (4b) as

\[ C_s(\tau, \rho) = \sum_{i=1}^{2r-s} \alpha_i \exp \{ j(\theta_{1,i} \tau + \theta_{2,i} \rho) \}, \tag{5} \]

where, in view of Remark R1, \( \theta_{1,i} = \theta_{2,i} = \omega_{1,i} \) for \( 1 \leq i \leq s \); \( \theta_{1,i} = \omega_{1,i} \) and \( \theta_{2,i} = \omega_{2,i} \) for \( s + 1 \leq i \leq r \); \( \theta_{1,i} = \omega_{2,i-r+s} \) and \( \theta_{2,i} = \omega_{1,i-r+s} \) for \( r + 1 \leq i \leq 2r - s \).

This relationship is more conveniently expressed in terms of ordered frequency pairs as

\[ (\theta_{1,i}, \theta_{2,i}) = (\omega_{1,i}, \omega_{2,i}), \quad i = 1, \ldots, r \tag{6a} \]

and

\[ (\theta_{1,i}, \theta_{2,i}) = (\omega_{2,i-r+s}, \omega_{1,i-r+s}), \quad i = r + 1, \ldots, 2r - s \tag{6b} \]

Furthermore, the \( \alpha_i \)'s are given by

\[ \alpha_i = \begin{cases} 2\lambda_i, & 1 \leq i \leq s, \\
\lambda_i, & s + 1 \leq i \leq r, \\
\lambda_{i-r+s}, & r + 1 \leq i \leq 2r - s. \end{cases} \tag{6c} \]

In view of assumption (A3), it may be seen that the ordered pairs \((\theta_{1,i}, \theta_{2,i})\), \( 1 \leq i \leq 2r - s \), are all distinct. The set \\{\((\theta_{1,i}, \theta_{2,i}); 1 \leq i \leq 2r - s\)\} consists precisely of all the ordered pairs which can be obtained from the \( r \) QC pairs by interchanging the individual components.

The objective of the quadratic coupling estimation problem is to extract the quadratically coupled frequency pairs \((\omega_{1,i}, \omega_{2,i}), \ i = 1, \ldots, r \), from a finite number of lags of \( \{C_x(\tau, \rho)\} \). Apart from this, it is also of interest to estimate the associated skewness values \( \lambda_i \). It may be noted that this problem is equivalent to obtaining the ordered frequency pairs \((\theta_{1,i}, \theta_{2,i}), 1 \leq i \leq r \), and the corresponding coupling strengths \( \alpha_i, 1 \leq i \leq r \), which are precisely impulse strengths at the locations \((\theta_{1,i}, \theta_{2,i}), 1 \leq i \leq r \), in the bispectrum of \( s(n) \). These parameters are precisely bispectral parameters of the signal process \( s(n) \). It can also be seen from (6c) that the coupling strength at a specific frequency pair location \((\Omega_1, \Omega_2)\) in the bispectrum domain equals the associated skewness, or twice the associated skewness, according as \( \Omega_1 \neq \Omega_2 \) or \( \Omega_1 = \Omega_2 \).

The equivalence of this general problem to the one formulated in Section 2.1 is amply clear once we note that the conjugate symmetric and antisymmetric versions \((C_x(\tau, \rho) + C_x^\ast(-\tau, -\rho))/2 \) and \((C_x(\tau, \rho) - C_x^\ast(-\tau, -\rho))/2j \) of the cumulants in Section 2.1 have the same structure as \( C_s(\tau, \rho) \) defined by (4a) and (5). (It is required to construct these conjugate symmetric and antisymmetric versions, since the \( \alpha_i \)'s of Section 2.1 can be complex, while the \( \alpha_i \)'s of this section are real.) In view of this equivalence, we note that the solutions proposed here for the estimation of quadratic phase coupling will also be applicable to the more general problem of estimating the bispectral parameters of a third-order stationary harmonic process corrupted by third-order white noise. (For example, if some peaks in the bispectrum arise out of difference interactions rather than the sum interactions considered above, then these will also be detected and quantified.) Our motivation here in working with the quadratic coupling model, rather than with the general one, is the physical interpretation associated with the former: The location of the bispectral peaks in the 2-D frequency plane can be interpreted as the QC pairs, while the amplitudes of the peaks can be interpreted in terms of the skewness associated with the QC pairs or, alternately, as the coupling strengths.

In the following sections, we develop a MUSIC-like algorithm, based on a matrix constructed using the third-order cumulants \( \{C_x(\tau, \rho)\} \), for obtaining the QC pairs \( \{(\omega_{1,i}, \omega_{2,i})\} \).

### 3. Construction of a complex cumulant matrix

The objective of this section is to construct a matrix from the complex third-order cumulants
of \{x(n)\} having a complete orthogonal eigenstructure. Based on this eigenstructure, we decompose its column space into two orthogonal subspaces, the signal subspace and the noise subspace. The structures of these subspaces are later exploited to construct estimators of the QC pairs. It is interesting to note that the eigenstructure properties of this cumulant matrix, which are used in the construction of the pseudo-bispectrum estimators, are very similar to those of the autocorrelation matrix that are exploited in the construction of the well-known MUSIC pseudo-spectral estimator [8].

Let \(N\) be a positive integer greater than \(q\), and let \(M = N^2\). Denote by \(C_x\) and \(C_s\) the \(M \times M\) matrices whose \((Ni + k + 1, Nj + m + 1)\)th entries are \(C_x(i - j, k - m)\) and \(C_s(i - j, k - m)\), respectively, where \(0 \leq i, j, k, m \leq N - 1\). It is interesting to note that these matrices can be expressed compactly in the following form:

\[
C_x = \sum_{0 \leq i, j \leq N - 1, 0 \leq k, m \leq N - 1} C_x(i - j, k - m)(u_i \otimes u_k)(u_j \otimes u_m)^H
\]

and

\[
C_s = \sum_{0 \leq i, j \leq N - 1, 0 \leq k, m \leq N - 1} C_s(i - j, k - m)(u_i \otimes u_k)(u_j \otimes u_m)^H,
\]

where \(u_i\) denotes the \(N \times 1\) column vector, whose \(i\)th entry equals 1 and whose other entries are all 0. The symbol \(\otimes\) denotes the standard Kronecker product. For vectors, \(x = [x_1, \ldots, x_n]^T\) and \(y = [y_1, \ldots, y_m]^T\), the Kronecker product is defined by \(z = x \otimes y\), where

\[
z = [x_1 y_1, x_1 y_2, \ldots, x_1 y_m, x_2 y_1, \ldots, x_2 y_m, \ldots, x_n y_1, \ldots, x_n y_m]^T.
\]

Equivalently, the \((m(i - 1) + j)\)th entry of \(z\) equals \(x_i y_j\), where \(1 \leq i \leq n, 1 \leq j \leq m\).

Similarly, for matrices \(X = [x_{ij}] \in \mathbb{C}^{N \times p}\) and \(Y = [y_{ij}] \in \mathbb{C}^{N \times q}\), the Kronecker product is defined by \(Z = X \otimes Y\), where the \((M(i - 1) + k, Q(j - 1) + m)\)th entry of \(Z\) equals \(x_{ij} y_{km}\).

It can be seen that \(C_x\) is an \(N^2 \times N^2\) block structured matrix constructed using \(N \times N\) blocks, such that its \((i, j)\)th block equals the \(N \times N\) Toeplitz matrix \(C_x(i - j)\), whose \((k, m)\)th entry is given by \(C_x(i - j, k - m)\). This can be pictorially represented as follows:

\[
C_x = \\
\begin{bmatrix}
C_x(0) & C_x(-1) & \cdots & C_x(-N + 1) \\
C_x(1) & C_x(0) & \cdots & C_x(-N + 2) \\
\vdots & \vdots & \ddots & \vdots \\
C_x(N - 1) & \cdots & \cdots & C_x(0)
\end{bmatrix},
\]

where

\[
C_x(i - j) = \\
\begin{bmatrix}
C_x(i - j, 0) & C_x(i - j, -1) & \cdots & C_x(i - j, -N + 1) \\
C_x(i - j, 1) & C_x(i - j, 0) & \cdots & C_x(i - j, -N + 2) \\
\vdots & \vdots & \ddots & \vdots \\
C_x(i - j, N - 1) & \cdots & \cdots & C_x(i - j, 0)
\end{bmatrix}
\]

A similar description applies to \(C_s\). In particular, we see that both \(C_x\) and \(C_s\) are Toeplitz–Block–Toeplitz matrices. The motivation for constructing the matrix \(C_x\) is derived from the structure of the autocorrelation matrix: If \(R_x(t)\) is the autocorrelation function of a wide-sense stationary process \(x(n)\), then the autocorrelation matrix for the first \(N\) samples of \(x(n)\) is the Toeplitz matrix \([R_x(i - j)]\). To extend this construction to third-order cumulants, it is natural to arrange the cumulants \(C_x(t, \rho)\) so as to yield a Toeplitz–Block–Toeplitz matrix and this is precisely what has been done here. The rank–subspace properties of \(C_x\), which we shall soon derive, provide further justification for constructing such a matrix.

Define the following vectors and matrices:

\[
e(\theta) = [1 \exp(j\theta) \ldots \exp(j(N - 1)\theta)]^T, \quad \theta \in [0, 2\pi),
\]

\[
a_i = e(\theta_{i,1}) \otimes e(\theta_{i,2}), \quad 1 \leq i \leq q,
\]

\[
A = [a_1, \ldots, a_q],
\]

\[
D = \text{diag}[a_1, \ldots, a_q].
\]

Then from (4a) and (5), we obtain the following matrix equations for \(C_x\) and \(C_z\), which show their explicit structure:

\[
C_x = ADA^H,
\]

\[
C_z = C_x + \gamma_w I_N \otimes I_K \equiv C_x + \gamma_w I_M.
\]

The following result shows that the 'signal subspace', which is defined as the subspace spanned by
the columns of $A$, or equivalently the range space of $A$, has dimension $q$, i.e., equal to the number of bispectral frequency pairs $(\theta_{1,i}, \theta_{2,i})$.

**Theorem 1.** The columns of $A$ are linearly independent, i.e., $\text{Rank}(A) = q$.

**Proof.** Introduce the set $\Xi_0 = \bigcup_{i=1}^{q} \{\theta_{1,i}\}$. It is readily verified that $\Xi_0$ also equals $\bigcup_{i=1}^{q} \{\theta_{2,i}\}$. In fact $\Xi_0$ is precisely the set of all the coupled frequency components, namely, $\bigcup_{i=1}^{q} \{\{\omega_{1,i}\} \cup \{\omega_{2,i}\}\}$.

Further, let $\Xi = \{(\theta_{1,i}, \theta_{2,i}): 1 \leq i \leq q\}$. We note that $\Xi$ is a subset of $\Xi_0 \times \Xi_0$. For any set $\Theta$, let $\#(\Theta)$ denote its cardinality, i.e., the number of elements in it. Since $\#(\Xi_0) \leq q < N$, it follows from a well-known result in linear algebra on Vandermonde vectors that the set $\Gamma_0 = \{ e(\theta) : \theta \in \Xi_0 \}$ is a linearly independent set in $\mathbb{C}^N$. By making use of the result that the Kronecker products between linearly independent sets of vectors are themselves linearly independent, it follows from the above observation that the set $\Gamma_0 \otimes \Gamma_0 = \{ e(\theta_1) \otimes e(\theta_2): (\theta_1, \theta_2) \in \Xi_0 \times \Xi_0 \}$ is a linearly independent set in $\mathbb{C}^{N^2}$. However, since $\Xi$ is a subset of $\Xi_0 \times \Xi_0$, it follows that the set $\Gamma$ defined by $\{ e(\theta_1) \otimes e(\theta_2): (\theta_1, \theta_2) \in \Xi \}$ is a subset of $\Gamma_0 \otimes \Gamma_0$. Therefore, $\Gamma$ is linearly independent in $\mathbb{C}^{N^2}$. Noting that the members of $\Gamma$ are precisely the vectors $a_i$, $1 \leq i \leq q$, which comprise the columns of $A$, the result follows.

A straightforward consequence of (9a), Theorem 1 and the fact that $D$ is a nonsingular $q \times q$ matrix is the following result, which gives the rank and subspace structure of the noiseless cumulant matrix $C_x$.

**Theorem 2.** (a) $C_x$ is a Hermitian matrix of rank $q$. (b) $\text{Range}(C_x) = \text{Range}(A)$. (c) $\text{Nullspace}(C_x) = \text{Nullspace}(A^H) \equiv \text{Range}^\perp(A)$.

In the development of the 2-D MUSIC algorithm, we shall be concerned with the eigensstructure of the cumulant matrix $C_x$. Theorem 2 immediately tells us what this eigensstructure looks like.

Since $C_x$ is an $M \times M$ Hermitian matrix of rank $q$, it has a set $\{v_1, \ldots, v_M\}$ of orthogonal eigenvectors such that $\{v_1, \ldots, v_q\}$ correspond to nonzero eigenvalues, say $\{\mu_1, \ldots, \mu_q\}$, and the set $\{v_{q+1}, \ldots, v_M\}$ is associated with the zero eigenvalue:

\[
\langle v_i, v_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq M. \tag{10a}
\]

\[
C_x v_i = \mu_i v_i, \quad \mu_i \neq 0, \quad 1 \leq i \leq q. \tag{10b}
\]

\[
C_x v_i = 0, \quad q + 1 \leq i \leq M. \tag{10c}
\]

It follows that $C_x$ has the eigendecomposition

\[
C_x = \sum_{i=1}^{q} \mu_i v_i v_i^H + \sum_{i=q+1}^{M} v_i v_i^H. \tag{11}
\]

Since, in particular, $\{v_1, \ldots, v_M\}$ is a basis for $\mathbb{C}^N$, it is clear from (10a) and (10b) that $\{v_1, \ldots, v_q\}$ is an orthonormal basis for $\text{Range}(C_x)$, which from Theorem 2 is the same as $\text{Range}(A)$. This $q$-dimensional subspace of $\mathbb{C}^N$ shall be referred to as the signal subspace. Likewise, from (10a), (10b) and the fact that $\{v_1, \ldots, v_M\}$ is a basis for $\mathbb{C}^N$, it follows that $\{v_{q+1}, \ldots, v_M\}$ is an orthonormal basis for $\text{Nullspace}(C_x)$ which is identical to the following three subspaces: (a) $\text{Range}^\perp(C_x)$, (b) $\text{Nullspace}(A^H)$, (c) $\text{Range}^\perp(A)$. This $(M - q)$-dimensional subspace of $\mathbb{C}^N$ shall be called the noise subspace.

From (9b), (10b) and (10c), it is clear that

\[
C_x v_i = \nu_i v_i, \quad 1 \leq i \leq q. \tag{12a}
\]

\[
C_x v_i = \gamma_i v_i, \quad q + 1 \leq i \leq M. \tag{12b}
\]

where $\nu_i = \mu_i + \gamma_i$, $1 \leq i \leq q$.

Equivalently, from (9b) and (11), we can write the eigendecomposition of $C_x$ as

\[
C_x = \sum_{i=1}^{q} \nu_i v_i v_i^H + \sum_{i=q+1}^{M} \gamma_i v_i v_i^H. \tag{13}
\]

Thus, $v_1, \ldots, v_q$ are eigenvectors of $C_x$ associated with the eigenvalues $\nu_1, \ldots, \nu_q$, and $v_{q+1}, \ldots, v_M$ are eigenvectors of $C_x$, all corresponding to the eigenvalue $\gamma_i$. We now assume that $|\nu_i| > |\gamma_i|$, $1 \leq i \leq q$. This assumption is plausible if $|\mu_i| > |\gamma_i|$, $1 \leq i \leq q$. It is intuitively clear that the latter can be safely assumed provided that $|\nu_i| > |\gamma_i|$, $1 \leq i \leq q$. It may be noted that this is equivalent to assumption (A5). As mentioned there, if, in particular, the noise has a symmetric distribution, then $\gamma_i = 0$, and this condition indeed holds good.
With this assumption, $\gamma_w$, which is an eigenvalue of $C_x$ of multiplicity $M - q$, can be determined as its eigenvalue of minimum magnitude. The eigenvectors $v_1, \ldots, v_q$ of $C_x$ shall be referred to as the signal eigenvectors of $C_x$ and their associated eigenvalues $\gamma_1, \ldots, \gamma_q$ as the signal eigenvalues. The vectors $v_{q+1}, \ldots, v_M$ shall similarly be called the noise eigenvectors of $C_x$ and the corresponding eigenvalue $\gamma_w$ the noise eigenvalue. This terminology is standard as far as eigenvector methods for harmonic retrieval using correlation matrices are concerned [8].

The above assumption, as we noted, facilitates the determination of the noise eigenvalue, and hence the separation of the signal and noise subspaces, or, equivalently, the determination of the signal eigenvalues and eigenvectors, directly from $C_x$.

At this stage, it is convenient to introduce the following matrices associated with the signal and noise eigenvectors:

$$E_s = [v_1, \ldots, v_q],$$

$$E_n = [v_{q+1}, \ldots, v_M].$$

Further, define the unitary $M \times M$ matrix

$$E = [E_s | E_n],$$

whose columns are the signal eigenvectors followed by the noise eigenvectors. In addition, we define

$$A_s = \text{diag}[\mu_1, \ldots, \mu_q],$$

$$A_n = \text{diag}[\Lambda_n | \mathbf{0}],$$

where, in (15b), $\mathbf{0}$ denotes the $(M - q) \times (M - q)$ zero matrix.

The eigendecompositions (11) and (13) of $C_s$ and $C_x$ may be compactly expressed in terms of these matrices as follows:

$$C_s = EA_s E_s^H = E_s A_s E_s^H,$$

and

$$C_x = E(A + \gamma_w I_M) E^H$$

$$\equiv E_s (A_s + \gamma_w I_q) E_s^H + \gamma_w E_n E_n^H. \tag{16b}$$

4. Generalized eigensubspace estimators

The following result gives a necessary and sufficient condition for an ordered pair in $[0, 2\pi) \times [0, 2\pi)$ to belong to the set $\Xi$ in terms of the signal subspace. This is the crucial result on which the construction of the generalized bispectrum estimators and, in particular, the MUSIC pseudo-bispectrum is based.

**THEOREM 3.** Let $\theta_1, \theta_2 \in [0, 2\pi)$. The ordered pair $(\theta_1, \theta_2)$ then belongs to $\Xi$ iff $e(\theta_1) \otimes e(\theta_2)$ belongs to the signal subspace, namely, $\text{Range}(A)$.

**PROOF.** The 'only if' part is trivial, since if $(\theta_1, \theta_2) \in \Xi$, then $\theta_1 = \theta_{1,i}$ and $\theta_2 = \theta_{2,i}$, for some $i = 1, \ldots, q$, whence $e(\theta_1) = e(\theta_{1,i})$ and $e(\theta_2) = e(\theta_{2,i})$ and, therefore, $e(\theta_1) \otimes e(\theta_2) = e(\theta_{1,i}) \otimes e(\theta_{2,i}) = \alpha_i \in \text{Range}(A)$.

The proof of the 'if' part is more involved: Let $e(\theta_1) \otimes e(\theta_2) \in \text{Range}(A)$. Then, we can write

$$e(\theta_1) \otimes e(\theta_2) = \sum_{i=1}^q c_i e(\theta_{1,i}) \otimes e(\theta_{2,i}), \tag{17}$$

where $c_1, \ldots, c_q$ are some complex constants.

Taking the inner product on both sides with $z_1 \otimes z_2$, where $z_1$ and $z_2$ are arbitrary vectors in $\mathbb{C}^N$, we get

$$\langle e(\theta_1), z_1 \rangle \langle e(\theta_2), z_2 \rangle = \sum_{i=1}^q c_i \langle e(\theta_{1,i}), z_1 \rangle \langle e(\theta_{2,i}), z_2 \rangle. \tag{18}$$

Since $z_2$ is chosen arbitrarily, it follows that

$$\langle e(\theta_1), z_1 \rangle \rangle e(\theta_2) = \sum_{i=1}^q c_i \langle e(\theta_{1,i}), z_1 \rangle e(\theta_{2,i}). \tag{19}$$

Further, since the set $\{\theta_2\} \cup \Xi_0 \equiv \{e(\theta): \theta \in \{\theta_2\} \cup \Xi_0\}$, is a linearly independent set in $\mathbb{C}^N$. Therefore, (19) implies that

$$\langle e(\theta_1), z_1 \rangle = \sum_{i \in \{\theta_{2,i}\}} c_i \langle e(\theta_{1,i}), z_1 \rangle, \tag{20}$$
which is equivalent to the equation
\[ \langle e(\theta_1), z_1 \rangle = \sum_{i: \theta_{1,i} = \theta_2} c_i \langle e(\theta_{1,i}), z_1 \rangle. \] (21)

Once again, in view of the arbitrariness of \( z_1 \), it follows from (21) that
\[ e(\theta_1) = \sum_{i: \theta_{1,i} = \theta_2} c_i e(\theta_{1,i}). \] (22)

Noting once again that the set \( \{e(\theta_{1,i})\} \cup \Gamma_0 \) is linearly independent, it follows from (22) that
\[ 1 = \sum_{i: \theta_{1,i} = \theta_2, \theta_{1,i} \neq \theta_1} c_i. \] (23)

Since \( \theta_{1,i}, \theta_{1,i} \), \( i = 1, \ldots, q \), are all distinct ordered pairs, it follows from (23) that \( \theta_{1,i} = \theta_{1,i} \) for some \( i = 1, \ldots, q \) and \( c_i = 1, c_j = 0, j \neq i \). In particular, \( (\theta_1, \theta_2) \in \Xi \). This completes the proof. \( \square \)

This result states that there is a one-to-one correspondence between the coupled frequency pairs and those vectors in the signal subspace which can be expressed as Kronecker products between steering vectors. In particular, this leads to a signal subspace interpretation for the quadratically coupled pairs. By noting that the noise subspace is the orthogonal complement of the signal subspace, an equivalent noise subspace interpretation could be given. The following corollary makes this more precise. This interpretation, as we shall soon see, leads us immediately to the class of generalized pseudo-bispectrum estimators.

**COROLLARY 1.** The ordered frequency pair \((\theta_1, \theta_2)\) in \([0, 2\pi) \times [0, 2\pi)\) belongs to the set \(\Xi\) iff \( E_{\theta_1}^H(e(\theta_1) \otimes e(\theta_2)) = 0\), i.e., iff \( e(\theta_1) \otimes e(\theta_2) \) is orthogonal to the noise subspace.

**PROOF.** This is a straightforward consequence of Theorem 3 and the fact that the noise subspace, namely the column space of \( E_{\theta_2} \), is the orthogonal complement of the signal subspace. \( \square \)

This corollary enables us to formulate search functions which would exhibit peaks precisely at the points \( \{(\theta_{1,i}, \theta_{1,i})\mid 1 \leq i \leq q\} \), namely at those and only those points in the two-dimensional frequency plane, where the true bispectrum also shows peaks.

**COROLLARY 2.** Let \( g(x_1, \ldots, x_{M-1}) \) be a function defined for all \( x_1, \ldots, x_{M-1} \) in \([0, \infty)\). Assume that \( g \) satisfies the following two properties: (a) \( g(x_1, \ldots, x_{M-1}) \geq 0 \), for all \( x_1, \ldots, x_{M-1} \), and (b) \( g(x_1, \ldots, x_{M-1}) = 0 \) if and only if \( x_1 = \cdots = x_q = 0 \). Then the 2-D function given by
\[ h(\theta_1, \theta_2) = 1/g((e(\theta_1) \otimes e(\theta_2)) E_{\theta_1}^H v_{q+1}, \ldots, (e(\theta_1) \otimes e(\theta_2)) E_{\theta_1}^H v_M) \] (24)
peaks precisely at the points \( (\theta_1, \theta_2) \in \Xi \).

**PROOF.** This is a direct consequence of the construction of \( g \) and Corollary 1. \( \square \)

It is clear from the above result that the QC pairs \( \{(\omega_{1,i}, \omega_{1,i})\} \) can be obtained via the peaks of the function \( h(\theta_1, \theta_2) \). If the noise eigenvectors are accurately estimated (which in turn depends on the accurate estimation of \( C_\tau \)), then for every peak of \( h(\theta_1, \theta_2) \) corresponding to the point \( (\theta_{1,i}, \theta_{1,i}) \), there exists another symmetrically located peak at \( (\theta_{2,i}, \theta_{1,i}) \). This is of course a consequence of the symmetry of the bispectrum about the \( \Omega_1 = \Omega_2 \) line. To ensure that this property holds good even when the noise vectors may not be estimated very accurately, we may select the function \( g \) such that \( h(\theta_1, \theta_2) = h(\theta_2, \theta_1) \).

Another method to do this is to symmetrize \( h \) by considering the search function
\[ h_1(\theta_1, \theta_2) = h(\theta_1, \theta_2) + h(\theta_2, \theta_1). \] (25a)

Alternately, we may find the peaks of the symmetric search function
\[ h_2(\theta_1, \theta_2) = \frac{1}{h(\theta_1, \theta_2) + 1/h(\theta_2, \theta_1)}. \] (25b)

The function \( h_1(\theta_1, \theta_2) \) given by (25a) has the following property: \( h_1(\theta_1, \theta_2) = \infty \) if and only if either \( h(\theta_1, \theta_2) = \infty \), or \( h(\theta_2, \theta_1) = \infty \), or both. Further, if \( h_1(\theta_1, \theta_2) = \infty \) for some \( (\theta_1, \theta_2) \), then also \( h_1(\theta_2, \theta_1) = \infty \). Similarly, the function \( h_2(\theta_1, \theta_2) \) given by (25b) has the property that \( h_2(\theta_1, \theta_2) = \infty \) if and only if \( h(\theta_1, \theta_2) = \infty \).
Further, if \( h_2(\theta_1, \theta_2) = \infty \), then \( h_2(\theta_2, \theta_1) = \infty \).

The method of symmetrizing the peaks as described by (25a) and (25b) can be put in a more general setting as follows: Let \( f(x, y) \) be a non-negative, symmetric function defined on \( 0 \leq x, y \leq \infty \), i.e., \( f(x, y) \geq 0 \), \( f(x, y) = f(y, x) \) \( \forall 0 \leq x, y \leq \infty \). Further, let \( f(x, y) \) be a monotonic increasing function of \( x \) for any fixed value of \( y \), i.e., \( f(x, y) > f(x', y) \) for \( 0 \leq x' < x \leq \infty \), \( 0 \leq y \leq \infty \). The assumed symmetry of \( f \) implies that \( f \) will also be a monotonic increasing function of \( y \) for a fixed value of \( x \). Consider then the following two cases:

(a) \( f(x, y) = \infty \) if and only if either \( x = \infty \), \( y = \infty \), or both \( x = y = \infty \). Construct the function:

\[
h_3(\theta_1, \theta_2) = h(h(\theta_1, \theta_2), h(\theta_2, \theta_1)). \tag{26a}
\]

This function satisfies the same property as \( h_1 \) mentioned above. In fact, \( h_1 \) is a special case of \( h_3 \) obtained from (26a) by setting \( f(x, y) = x + y \).

(b) \( f(x, y) = \infty \) if and only if both \( x = y = \infty \). We again construct the function:

\[
h_4(\theta_1, \theta_2) = f(h(\theta_1, \theta_2), h(\theta_2, \theta_1)). \tag{26b}
\]

This function, then, satisfies the same property as \( h_2 \) mentioned above. In fact, \( h_2 \) is a special case of \( h_4 \) obtained from (26b) by setting \( f(x, y) = 1/(1/x + 1/y) \).

From these discussions, it is clear that, apart from symmetrizing the location of peaks, \( h_3 \) considers the value of \( h \) at either \((\theta_1, \theta_2)\) or \((\theta_2, \theta_1)\) in deciding whether or not to exhibit a peak at the locations \((\theta_1, \theta_2)\) and \((\theta_2, \theta_1)\). However, \( h_4 \), apart from symmetrizing the location of peaks, considers the value of \( h \) at both \((\theta_1, \theta_2)\) and \((\theta_2, \theta_1)\) with equal weightage in deciding whether or not to exhibit a peak at the locations \((\theta_1, \theta_2)\) and \((\theta_2, \theta_1)\). Which function better estimates the QC pairs is not clear, however.

5. The MUSIC pseudo-bispectrum estimator

The bispectrum MUSIC estimator is obtained by choosing \( g(x_1, \ldots, x_{M-3}) = \sum_{i=1}^{M-3} x_i^2 \) and using the symmetric search function \( h_2(\theta_1, \theta_2) \) defined by (25b). It is easy to verify after some minor simplifications that the bispectrum MUSIC estimator can be expressed compactly in terms of the signal eigenvectors as

\[
h_{B-MUSIC}(\theta_1, \theta_2) = 1/(2M - P(\theta_1, \theta_2) - P(\theta_2, \theta_1)), \tag{27a}
\]

where

\[
P(\theta_1, \theta_2) = \| E^H \{(e(\theta_1) \otimes e(\theta_2)) \} \|^2. \tag{27b}
\]

We term \( h_{B-MUSIC} \) as the MUSIC pseudo-bispectrum.

The peaks of the MUSIC pseudo-bispectrum are symmetrically located, i.e., if a peak occurs at \((\Omega_1, \Omega_2)\), then it also occurs at \((\Omega_2, \Omega_1)\). Moreover, the computation of this estimator, according to (27), involves only the signal eigenvectors, which, in practice, are much fewer than the noise eigenvectors, and are generally more accurately estimated. Theoretically, \( s \) peaks are expected to lie on the diagonal line \( \Omega_1 = \Omega_2 \), while \( r - s \) peaks may be expected to lie on either side of this line, symmetrically arranged in pairs about this line. In practice, the \( r \) QC pairs may be obtained as the peaks lying in the region \( \Omega_1 \geq \Omega_2 \). Some of the true peaks lying on the line \( \Omega_1 = \Omega_2 \) may split due to finite data length effects. In this case, we will still have \( r \) peaks in the region \( \Omega_1 \geq \Omega_2 \). This is because if a peak, theoretically present at \((\omega_0, \omega_0)\), splits, then one part goes to the region \( \Omega_1 > \Omega_2 \), while another to the region \( \Omega_2 > \Omega_1 \) by virtue of the symmetry of \( h_{B-MUSIC} \) about the line \( \Omega_1 = \Omega_2 \). Hence, no error due to splitting can occur in the estimation of \( r \). This is one of the reasons for constructing a symmetric estimator.

6. Simulation results

In this section, we present the results of some simulation studies on the MUSIC pseudo-bispectrum, performed using computer-generated data.

The noise \( w(n) \) is generated by passing zero-mean white Gaussian noise of unit variance through a quadratic nonlinearity given by \( y = ax^2 + bx - a \), where \( a \) and \( b \) are some constants. It can be seen that the noise generated is zero-mean and has a variance \( \sigma_w^2 = E\{w^2(n)\} \)
\[ x^2 + 2ax + a^2 \] and a skewness \( \gamma = E \{ w^3(n) \} = 8a^3 + 6ab^2 \).

The method used for estimating the cumulant matrix \( C_\tau \) is briefly outlined below. Further details on the estimation of third-order cumulants are available in [14].

Let \( x(0), x(1), \ldots, x(L - 1) \) be a realization of a finite length of data. Then we proceed as follows:

(1) Segment the data into \( K \) records of \( N' \) samples each so that \( L = KN' \).

(2) Subtract the average values of each record from the data set contained in that record to make the sample mean zero.

(3) Let

\[ x_n^*(k) = x^*(i - 1)N' + k, \]

\( i = 1, \ldots, K, \quad k = 0, 1, \ldots, N' - 1, \]

where

\[ x^*(i - 1)N' + k = x((i - 1)N' + k) - (1/N') \sum_{k=0}^{N'-1} x((i - 1)N' + k). \]

(4) Estimate the third moment (or cumulant) sequence

\[ C^{(3)}(\tau, \rho) = (1/s_2 - s_1) \times \sum_{n=s_1}^{s_2} \left[ x^{(3)}(n)x^{(3)}(n + \tau)x^{(3)}(n + \rho) \right], \]

where \( s_1 = \max[0, -\tau, -\rho] \) and \( s_2 = \min[N' - 1, N' - 1 - \tau, N' - 1 - \rho] \).

Clearly, the range of admissible values of \( \tau, \rho \) in the above expression is given by

### Table 1

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<th>No. of samples per record</th>
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<th>Noise eigenvalue of largest magnitude</th>
<th>Mean noise eigenvalue</th>
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</tr>
<tr>
<td>5</td>
<td>256</td>
<td>64</td>
<td>790,792</td>
<td>0.28</td>
<td>0.10</td>
<td>0.14</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Serial no.</th>
<th>No. of records</th>
<th>No. of samples per record</th>
<th>Mean of QC pair estimates ((\alpha_1, \alpha_2, \gamma_1))</th>
<th>Std.dev. of QC pair estimates ((\sqrt{\text{Var}(\alpha_1)}, \sqrt{\text{Var}(\alpha_2)}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>64</td>
<td>64</td>
<td>(3.16860, 4.9213, -0.0530, 0.1406)</td>
<td>(0.0013, 0.0019)</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>64</td>
<td>(2.9901, 4.9970, 0.0082, 0.00461)</td>
<td>(0.0013, 0.0048)</td>
</tr>
<tr>
<td>3</td>
<td>96</td>
<td>64</td>
<td>(2.9902, 5.0093, 0.0067, 0.0072)</td>
<td>(0.0013, 0.0048)</td>
</tr>
<tr>
<td>4</td>
<td>128</td>
<td>64</td>
<td>(3.0082, 4.9770, 0.0013, 0.0048)</td>
<td>(0.0001, 0.0019)</td>
</tr>
<tr>
<td>5</td>
<td>256</td>
<td>64</td>
<td>(3.0013, 5.0032, 0.0001, 0.0019)</td>
<td>(0.0013, 0.0048)</td>
</tr>
</tbody>
</table>
Since the desired range of $\tau, \rho$ required to form the matrix $C_\chi$ is
$$(N' - 1) \leq \tau, \rho \leq (N' - 1),$$
Since $N'$ must be chosen sufficiently larger than $N$. This will guarantee that at least one term will contribute to the above summation.

(5) Average $C_{\chi}^\mu(\tau, \rho)$ over all the $K$ segments:
$$C_{\chi}(\tau, \rho) = (1/K) \sum_{k=1}^{K} C_{\chi}^\mu(\tau, \rho).$$

(6) Construct the $N^2 \times N^2$ matrix $C_\chi$ whose $(Ni + k + 1, Nj + m + 1)$th entry equals $C_{\chi}(i - j, k - m)$, with $i, j, k, m$ taking integer values from 0 to $N - 1$. Replace $C_{\chi}$ by its Hermitian part $(C_{\chi} + C_{\chi}^\dagger)/2$.

The results of our simulation studies are presented in Tables 1 and 2 and Figs. 1–4. These are briefly discussed below.

In the first simulation example, we choose the following signal parameters. The total number of sinusoids (p) is chosen as 6. There is exactly one quadratically coupled frequency pair in the signal $(r = 1)$ given by $(\omega_{1,1}, \omega_{2,1}) = (2\pi/10)(3, 5) = (1.885, 3.1416)$ in radians. The frequency in the signal through which these two are quadratically coupled is $\omega_{3,1} = \omega_{1,1} + \omega_{2,1} = 2\pi \times 8/10 = 5.026$ radians. The amplitudes of the harmonic components at these frequencies are given by $A_{1,1} = A_{2,1} = A_{3,1} = \sqrt{10}$. Apart from these, there are $p - 3r = 3$ uncoupled harmonic components at (harmonically related) frequencies $\omega_4 = 2\pi \times 6/10$, $\omega_5 = 2\pi \times 8/10$ and $\omega_6 = \omega_4 + \omega_5 = 2\pi \times 14/10 = 2\pi \times 4/10 \equiv 2\pi \times 6/10 \pmod{2\pi}$. The amplitudes of these components are once again chosen as $A_4 = A_5 = A_6 = \sqrt{10}$. The additive noise $\{w(n)\}$ is obtained by passing zero-mean, unit variance, white non-Gaussian noise $\{v(n)\}$ through the memoryless quadratic nonlinearity given by $y = 0.1(x^2 - 1 + x)$. Thus, $w(n) = 0.1(v^2(n) - 1 + v(n))$. It can easily be verified that $\{w(n)\}$ is a zero-mean, third-order white non-Gaussian process with variance $\sigma_w^2 = \mathbb{E}\{w^2(n)\} = 0.03$ and skewness $\gamma_w = \mathbb{E}\{w^3(n)\} = 0.014$. The number $N$, which defines the size of the cumulant matrix $C_{\chi}$, is chosen as 5. Thus, the size of the cumulant matrix we are working with is $N^2 \times N^2 = 25 \times 25$.

Since $r = 1$ and $s = 0$, theoretically the number of signal eigenvalues is $2r - s = 2$. The number of noise eigenvalues is $25 - 2 = 23$ (all of which equal $\gamma_w$ in the ideal, given statistics case). Table 1 gives the evolution of the estimated signal eigenvalues, the mean noise eigenvalue, as well as the noise eigenvalue of maximum magnitude and the standard deviation of the noise eigenvalues, with increasing record length $K$. The noise eigenvalue of maximum magnitude, when compared with the signal eigenvalues, yields a measure of the separability of the signal and noise subspaces. The larger the magnitude of this noise eigenvalue, the harder it is to separate the signal and noise subspaces. As the record length $K$ increases, it is seen that the magnitude of the maximum magnitude noise eigenvalue decreases. More specifically, when the number of records is 64, we see that this noise eigenvalue has a magnitude comparable to the signal eigenvalues. In this case, the signal subspace dimension may be overestimated as 3 rather than 2. However, with increasing record length, it becomes clear that the signal subspace indeed has dimension 2. This confirms the presence of just one coupled frequency pair. The magnitude of the mean noise eigenvalue also shows a tendency to decrease, and so does the standard deviation of the noise eigenvalue. These observations suggest, in particular, that the noise eigenvalue estimates are consistent. The signal eigenvalues, on the other hand, do not exhibit much variation with the record length. Table 2 gives the corresponding evolution of the sample mean and variance of the QC pair estimates computed from the locations of the MUSIC pseudo-bispectrum.

Fig. 1. Signal and noise parameters as for Table 1. Number of records $(K) = 256$, number of samples per record $(N') = 64$. Each division on either axis $(\omega_1$ or $\omega_2)$ equals $2\pi/10$ radians.
Fig. 2. MUSIC pseudo-bispectrum plots. \((\omega_{1,i}, \omega_{2,i}) = (2\pi/10)(3, 5) = (1.885, 3.1416), (\omega_{1,2}, \omega_{2,2}) = (2\pi/10)(6, 8) = (3.7699, 5.0265), \omega_0 = 5.0872, \omega_0 = 5.3246\). (a) \(A_{i,i} = A_{i,2} = A_{2,1} = 1, i = 1, 2, A_{i} = 1, i = 3, 4\). SNR per sinusoid = 15 dB, skewness ratio per QC pair = 12 dB. Simulation performed for the ideal, infinite data length (or given statistics) case. Each division on either axis (\(\omega_1\) or \(\omega_2\)) equals \(2\pi/10\) radians. (b) Projection of Fig. 1(a) on the \((\omega_1, h)\) plane (or \((\omega_2, h)\) plane). The different line styles correspond to cross-sections of the 3-D plot along the grid lines parallel to the \((\omega_1, h)\) plane of Fig. 1(a) (or \((\omega_2, h)\) plane). Each division on the \(\omega_1\) (or \(\omega_2\)) axis equals \(2\pi/10\) radians. (c) Amplitudes are the same as those for Fig. 1(a). Number of records = \(K = 50\), number of samples per record = \(N' = 200\). Each division on either axis (\(\omega_1\) or \(\omega_2\)) equals \(2\pi/10\) radians. (d) Projection of Fig. 1(c) on the \((\omega_1, h)\) plane (or \((\omega_2, h)\) plane). Each division on the \(\omega_1\) (or \(\omega_2\)) axis equals \(2\pi/10\) radians. (e) \(A_{i,i} = A_{i,2} = A_{2,2} = 0.5, i = 1, 2, A_{i} = 1, i = 3, 4\). SNR per coupled sinusoid = 9 dB, SNR per uncoupled sinusoid = 15 dB, skewness ratio per QC pair = 6 dB. Number of records = \(K = 50\), number of samples per record = \(N' = 200\). Each division on either axis (\(\omega_1\) or \(\omega_2\)) equals \(2\pi/10\) radians. (f) Projection of Fig. 1(e) on the \((\omega_1, h)\) plane (or \((\omega_2, h)\) plane). Each division on the \(\omega_1\) (or \(\omega_2\)) axis equals \(2\pi/10\) radians.
peaks. It is evident that as the record length increases, there is a marked decrease in the variance and, hence, an improvement in the performance of the algorithm. (Of course, these observations are all consistent with the law of large numbers.) Fig. 1 shows a three-dimensional plot of the MUSIC pseudo-bispectrum obtained using $K = 256$ records and $N' = 64$ time samples per record. The presence of two peaks symmetrically located about the $\Omega_1 = \Omega_2$ line indicates the presence of just one coupled frequency pair. We observe that one of them is located approximately at $2\pi/10$ (3, 5) radians and the other one at $2\pi/10$ (5, 3) radians. The absence of any peak in the vicinity of $(\omega_4, \omega_5)$ indicates no phase coupling between these two frequencies, as expected. This testifies that the pseudo-bispectrum is insensitive to any set of three harmonically related frequencies that do not have quadratic phase relations.

Figs. 2-5 display some typical plots of the pseudo-bispectrum MUSIC estimator for different values of the signal parameters and the data length. The sharp peaks in all these plots illustrate that this estimator does not have high resolution. In these experiments, both the number of phase-coupled pairs $(r)$ and the number of uncoupled sinusoids, i.e., $p = 3r$, are chosen to be 2. Thus, the total number of sinusoids $(p)$ in the signal is 8. The non-Gaussian noise is generated as in the first experiment, and hence once again $N$ is chosen as 5.

The $x$, $y$ and $z$ axes in these plots are labeled as the $\omega_1$, $\omega_2$ and the $h$ axes, respectively. In some experiments, a plot of the ideal MUSIC pseudo-bispectrum estimator $h_B$MUSIC has also been shown. Such figures obviously correspond to the estimator obtained using an infinite data length. Moreover, for clarity, in some experiments, the projection of the 3-D plot of the estimator on the $(\omega_1, h)$ plane has also been shown. Since the estimator is symmetric about the $\omega_1 = \omega_2$ line, this also corresponds to the projection on the $(\omega_2, h)$ plane.

The following qualitative conclusions emerge from these figures.

1) As the number of records increases (i.e., the data length is increased), the peaks become more pronounced with respect to the background noise level. Compare Figs. 4(a) and 4(b) or Figs. 5(a) and 5(b).

---

Fig. 3. MUSIC pseudo-bispectrum plots. $(\omega_{1,1}, \omega_{1,2}) = (2\pi/10)(4, 4) = (2.513, 2.513)$, $(\omega_{2,1}, \omega_{2,2}) = (2\pi/10)(6, 8) = (3.77, 5.0265)$, $\omega_l = 5.0872$, $\omega_b = 5.3246$. (a) $A_{1,1} = A_{2,2} = 0.5$, $i = 1, 2$, $A_i = 1$, $i = 3, 4$. SNR per coupled sinusoid = 9 dB, SNR per uncoupled sinusoid = 15 dB, skewness ratio per QC pair = 6 dB. Number of records = $K = 50$, number of samples per record = $N' = 200$. Each division on either axis $(\omega_1$ or $\omega_2$) equals $2\pi/10$ radians. (b) A different realization for the same values of the signal parameters as well as the number of records and the number of samples per record as in Fig. 2(a). Each division on either axis $(\omega_1$ or $\omega_2$) equals $2\pi/10$ radians. (c) Projection of Fig. 1(b) on the $(\omega_1, h)$ plane $(\omega_2, h)$ plane. Each division on the $\omega_1$ (or $\omega_2$) axis equals $2\pi/10$ radians.
Fig. 4. MUSIC pseudo-bispectrum plots. \((\omega_{1,1}, \omega_{2,1}) = (2\pi/20)(4.6) = (1.2566, 1.88), (\omega_{1,2}, \omega_{2,2}) = (2\pi/20)(15.17) = (4.75, 5.3), \omega_{\phi} = 1.8988, \omega_{0} = 4.5996.\) (a) \(A_{1,i} = A_{2,i} = A_{3,i} = 2, i = 1, 2, A_{4} = 1, i = 3, 4.\) SNR per coupled sinusoid = 21 dB, SNR per uncoupled sinusoid = 15 dB, skewness ratio per QC pair = 18 dB. Number of records = \(K = 15,\) number of samples per record = \(N = 100.\) Each division on either axis (\(\omega_{1}\) or \(\omega_{2}\)) equals \(2\pi/20\) radians. (b) Amplitudes are the same as those for (a) and finite data length is used. Number of records = \(K = 10,\) number of samples per record = \(N = 100.\) Each division on either axis \((\omega_{1}\) or \(\omega_{2}\)) equals \(2\pi/20\) radians.

Fig. 5. MUSIC pseudo-bispectrum plots. \((\omega_{1,1}, \omega_{2,1}) = (2\pi/20)(15.8) = (4.72, 5.1), (\omega_{1,2}, \omega_{2,2}) = (2\pi/20)(13.6) = (4.08, 1.885), \omega_{\phi} = 6.2411, \omega_{0} = 5.2529.\) (a) \(A_{1,i} = A_{2,i} = A_{3,i} = 1, i = 1, 2, A_{4} = 1, i = 3, 4.\) SNR per coupled sinusoid = 15 dB, skewness ratio per QC pair = 12 dB. Simulation performed for ideal, infinite data length case. Each division on either axis \((\omega_{1}\) or \(\omega_{2}\)) equals \(2\pi/20\) radians. (b) Amplitudes are the same as those for (a) and finite data length is used. Number of records = \(K = 15,\) number of samples per record = \(N = 200.\) Each division on either axis \((\omega_{1}\) or \(\omega_{2}\)) equals \(2\pi/20\) radians.

(2) When the skewness of the signal triplet associated with a QC pair decreases for a fixed value of the noise skewness, the performance deteriorates, as is shown by comparing Figs. 2(c) and 2(e). This suggests that, in particular, when the noise statistics get closer to Gaussian, the signal to noise skewness ratio increases, and the performance of the estimator improves. This observation is consistent with the fact that, theoretically, the estimator improves. This observation is consistent with the fact that, theoretically, the estimator is insensitive to Gaussian components (colored or white) present in the noise.

(3) The amplitude of a peak does not give any indication of the skewness (or coupling strength) of the signal associated with the corresponding phase-coupled pair. In other words, for fixed values of the signal parameters, if the QC pairs are estimated using two different realizations of the process, although the locations of the peaks in the two realizations will not differ appreciably, the amplitude of these peaks will invariably differ. This becomes evident on comparing Figs. 3(a) and 3(b). In fact, as the theoretical development of the estimator shows, the amplitude of each pseudo-bispectral peak, in the ideal infinite data situation, is \(\infty.\)
(4) A comparison of the MUSIC pseudo-bispectrum with the AR parametric bispectrum estimator proposed by Raghveer and Nikias in [19] shows that the resolutions of these two estimators are comparable for the same order of data length and the signal to noise ratio. In this direction, the resolution shown by the pseudo-bispectrum plot of Fig. 5(b) can be compared to the results of [19]. The pseudo-bispectrum proposed here can, via simulation, be shown to have a far higher resolution than the conventional nonparametric Fourier based bispectrum estimator. However, plots showing these comparisons have not been included here due to space limitations.

(5) When two frequencies belonging to two different QC pairs are quite close, the resolution improves significantly as the number of records used to form the cumulant estimates increases. Thus, for the ideal case in which the data length is infinite, perfect resolution is achieved, and as the number of records decreases, merging of the peaks becomes more prominent (compare Figs. 5(a) and 5(b)).

(6) It is worth mentioning here that in order to form consistent estimates of the cumulants, it is necessary to perform time averaging as well as ensemble averaging (i.e., averaging over statistically independent records), because for certain values of the frequencies the harmonic signal may not be third-order ergodic [16, 19].

7. Conclusions

We have, in this paper, developed a MUSIC-like algorithm for estimating the bispectral parameters of harmonics in white noise and have focussed our attention on the quadratic phase coupling problem, having demonstrated the mathematical equivalence between the two problems.

A third-order cumulant matrix having a normal structure is constructed, and a result giving a one to one correspondence between the quadratically coupled frequency pairs, and the Kronecker product between steering vectors lying in its signal subspace is obtained. Based on this elegant algebraic result and the eigendecomposition of the cumulant matrix, a class of pseudo-bispectral estimators is constructed. We finally arrive at a special case of one of them, termed as the MUSIC pseudo-bispectrum, in view of its close structural similarity to the well-known MUSIC pseudo-spectral estimator which is based on the eigendecomposition of the autocorrelation matrix. Simulation results confirm the high-resolution performance of the MUSIC pseudo-bispectrum in detecting and resolving quadratically coupled frequency pairs.

We would like to conclude by mentioning some of the open issues in this work. An interesting problem to look at would be to develop information theoretic criteria for estimating the rank \( q = 2r - s \) of \( C_x \), or, equivalently, the number of signal eigenvalues, which also equals the number of bispectral peaks. It would also be worth evaluating the Cramer–Rao bounds for the variance of the QC pair estimates for different probability distributions of the noise process, and using these bounds as a yardstick to evaluate the performance of the MUSIC pseudo-bispectrum. Another problem of interest is to carry out a perturbation theoretic analysis of the MUSIC-like algorithm proposed in this paper to derive theoretical upper bounds on the estimation error variance, in terms of the number of data samples and the noise statistics.

8. References


