FREE DAMPED FLEXURAL VIBRATION ANALYSIS OF COMPOSITE CYLINDRICAL TUBES USING BEAM AND SHELL THEORIES

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An investigation is described of the free vibration characteristics, i.e., natural frequencies and damping ratios in flexural modes of cylindrical tubes made from fibre reinforced materials with potential usage as composite shafts. This paper is concerned with only the first circumferential modes, which are essentially the beam bending modes. The results obtained from beam and shell theories are compared, which leads to the determination of limits of tube parameters up to which beam theory gives valid results. Viscoelastic material damping has been assumed and system loss factors have been determined for various modes, by using the complex modulus approach in the beam and shell theories. The results obtained indicate that for practically used tube parameters, flexural frequency values obtained from beam theory with Timoshenko type shear deformation match well with those from shell theory without thickness shear deformation. The closeness of the results from the two theories improves further when the shell theory accounts for thickness shear deformation. In comparing the results obtained from beam theory and shell theory, emphasis is given to possible explanations of the points of difference in terms of physical behavior of the shell.

1. INTRODUCTION

The possible use of fibre reinforced composite cylindrical tubes as shafts has received much attention from researchers over the past decade. Testing and analysis has been carried out on composite driveshafts for helicopters and automobiles [1–3]. Apart from the direct benefits of weight savings, these have other indirect advantages such as a reduced number of bearings, better placement of critical speeds and smoother operation. Bauchau [1] formulated the problem for maximization of the critical speed while meeting the design requirements. Hetherington et al. [3] successfully conducted experiments on a composite helicopter power transmission shaft beyond its second critical speed, and solved the design problem for optimum placement of critical speeds.

In these analyses, equivalent modulus beam theory was used for dynamic analysis of the shafts. In this approach the equivalent longitudinal and in-plane shear moduli are determined by using classical laminate theory. These moduli are then used to find the flexural stiffness and shear stiffness of the tube. With these stiffness values, the natural frequency is found by using Timoshenko beam theory just as for isotropic materials. As a result of the simplifying assumptions, this procedure has some limitations which are discussed below. A detailed study of these limitations and the effect which they have on results is the primary motivation for the present work.

For a shaft with the same orientation $\pm \theta$, in all the layers, as is produced by the process of filament winding with a constant winding angle, one of the laminate axes of symmetry
coincides with the tube geometric axis. In such a case, the in-plane modulus can be used directly to calculate the flexural stiffness. However, for a case in which different layers have different fibre orientations, the in-plane modulus cannot be used directly for calculation of the flexural stiffness, even though the lamination scheme may be symmetric. The contribution of a particular layer also depends on its distance from the neutral axis (outer plies contribute more). However, the in-plane modulus is still used for calculation of the flexural stiffness, because if the tube is thin walled, then the second moment of area which is used in the expression for the flexural stiffness is approximately proportional to \( t \), the thickness of the tube wall. As a consequence of this approximation, the contribution from a particular angle can be attributed solely to the lamination thickness corresponding to that angle, irrespective of its position in the lamination scheme.

If the lamination sequence is asymmetric, bending/stretching coupling will be present which cannot be directly incorporated in the beam theory. Figure 1 shows the effects of bending/stretching coupling induced in an asymmetrically laminated tube. In the flexural mode of the tube as shown in Figure 1(a), the lower half portion of a particular layer is in tension while the upper half is in compression. Thus the direction of the stretching force induced at the middle surface of the tube wall is opposite in the two halves. Similarly, if pure stretching is applied, as shown in Figure 1(b), then the induced bending moment due to bending/stretching coupling will be symmetric about the cross-section and will not cause any flexure of the tube. Thus, owing to the regular geometry of the cross-section the bending/stretching coupling, although present in the laminate construction, may not affect the bending behavior.

In conventional beam theory analysis, it is also assumed that plane sections remain plane after bending and the shape of the cross-section is not deformed. However, in an actual case, shell theory results show that for some configurations the cross-sections also deform.

![Diagram](image)

**Figure 1.** The effect of bending/stretching coupling in an asymmetrically layered tube. (a) Stretching induced under pure bending; (b) bending moment induced under pure tension.
during bending, which can affect the bending frequencies significantly. In the case of thin tubes of short length, shell-type modes are present in the same range in which the flexural frequencies lie, and it is important to investigate whether they will affect the flexural frequencies and the shaft dynamic behavior because of couplings caused by curvature or anisotropy.

Another limitation of the equivalent modulus beam theory is in regard to the modelling of thickness shear deformation. The thickness shear of a composite tube involves both the in-plane modulus \( G_{xy} \) and the thickness shear modulus \( G_{xy} \) of the laminate; but in the calculation only the in-plane modulus is used. However, in shell theory, the two effects—i.e., in-plane shear and the thickness shear of the wall—can be separately accounted for.

Greenberg and Stavsky [4] have shown that, for a shell with particular \( t/R \) ratio, as \( L/R \) increases the circumferential mode number at which the minimum frequency occurs decreases. Beyond a particular value of \( L/R \), the flexural mode \( n = 1 \) invariably provides the fundamental frequency. The same phenomenon occurs if, for constant \( L/R \), \( t/R \) is increased. But even when shell-type modes provide the fundamental frequency, the flexural modes do exist, although at higher frequency values, and are predicted by the shell theory. To what extent the beam theory predictions match with the shell theory results needs to be analyzed.

In order to investigate the questions raised above, and analyze the difference in results of the two theories, it was required to find the flexural frequencies and system loss factors by using shell theory, accounting for all the effects, and to compare them with those obtained from the beam theory generally used for composite shafts.

The shell theory used in the present analysis is first order shear deformation theory for moderately thick shells [5]. In order to take into account the distribution of thickness shear, a shear correction factor is used. A large number of the analyses for shells available in the literature [4–10] vary in respect to the assumption made about the thickness shear distribution through the wall thickness. However, a number of results [9, 10] show that for thin or moderately thick shells the change in the thickness shear distribution does not make much difference in the frequency values, in particular the frequencies corresponding to flexural modes. Many of the analyses of anisotropic shells give results for orthotropic shells. Vanderpool and Bert [6] analyzed monoclinic shells under free–free boundary conditions by an exact, iterative analysis which included all the shear normal couplings and the thickness shear–shear coupling terms. In a similar way, the present shell analysis also accounts for these coupling effects, as well as the couplings arising due to asymmetry of lamination scheme.

2. BEAM THEORY

2.1. FORMULATION

The beam theory used to find the flexural frequencies and system damping values is the same as that usually employed for the analysis of composite shafts [1–3]. The equivalent longitudinal and in-plane shear stiffness of the laminate are found as reported by Tsai and Hahn [11] (a list of nomenclature is given in the Appendix):

\[
E = \frac{(4(U_1 - U_3)(U_1 + U_3) - \beta^2 U_3^2)}{U_1 - \beta U_2 + \gamma U_3},
\]

(1)
where $U_1$, $U_2$, $U_3$, $U_4$ and $U_5$ are laminate invariants,

$$
\gamma = \sum_{i=1}^{n} \frac{t_i}{t} \cos 4\alpha_i, \quad \beta = \sum_{i=1}^{n} \frac{t_i}{t} \cos 2\alpha_i.
$$

The equivalent shear modulus is

$$G = U_4 - U_3\gamma. \quad (2)$$

When finding the values of the equivalent $E$ and $G$, it is assumed that the axis of the tube coincides with a material symmetry axis, so that no couplings among different stress components take place. The only point of departure from reference [11] is that the orthotropic moduli are in complex form to take into account the material damping. The real part represents the storage modulus component and the imaginary part represents the loss modulus. The invariants $U_i$ are also complex. These equivalent stiffness terms are then used to set up the Lagrangian. Because of relatively low values of the shear modulus, the shear deformation in thin walled composite beams may be quite large and thus has to be taken into account when setting up the energy terms. It has been shown that using Euler–Bernoulli theory might give erroneous results [12]. It is to be noted that $G$ used here is the in-plane modulus and not the thickness shear modulus.

For calculation of the shear energy, Timoshenko-type shear deformation is assumed. Rotary inertia is also included in the kinetic energy terms. Transverse displacements due to bending ($w_t$) and shear effects ($w_s$) are assumed to be functions of the $x$ co-ordinate only (see Figure 2(a)). Since we are dealing with the free vibrations case, the actual displacements would be harmonic functions of time. However, when setting the derivatives of the Lagrangian with respect to the solution coefficients equal to zero in order to obtain the
solution equations, the time dependence of all terms in the final equations can be taken out. Hence, the Lagrangian function can be written as \( \tilde{L} = \tilde{U} - \tilde{T} \), where \( \tilde{U} \) and \( \tilde{T} \) are the amplitudes of the strain and kinetic energies respectively and are independent of time. Thus the Lagrangian is

\[
\tilde{L} = \int_0^L \left[ EI(\partial^2 w_0/\partial x^2)^2 + GK(\partial w_0/\partial x)^2 - \omega^2 TIw^2 - \omega^2 R\alpha^2 \right] dx,
\]

where \( EI \) is the flexural stiffness, \( = \pi R^3(1 + t^2/4R^2)E \), \( GK \) is the shear stiffness, \( = k'AG \), \( k' \) is the shear correction factor, \( = 1/2 \), \( TI \) is the transverse inertia, \( = 2\pi R^3(1 + t^2/4R^2) \), \( w \) is the total displacement, \( = w_r + w_s \), and \( \alpha \) is the rotation due to bending only, \( = \partial w_0/\partial x \). Thus,

\[
\partial w_0/\partial x = (\partial w/\partial x - \alpha).
\]

It may be noted that \( \tilde{L} \) in equation (3) is complex since \( E \) and \( G \) are in complex form. Also, for a composite shaft, the shear correction factor depends on the lamination scheme and the laminate moduli. However, it can be shown [14] that for hollow tubes of moderate thickness, an approximate value of 0.5 may be used.

2.2. SOLUTION EQUATIONS

Displacement functions are assumed for \( w \) and \( \alpha \) as

\[
w = \sum_{l=1}^N a_l w^l(x), \quad \alpha = \sum_{l=1}^N b_l \alpha^l(x).
\]

The solution equations are obtained by differentiating \( \tilde{L} \) in equation (3) with respect to the solution coefficients and setting the derivatives equal to zero: i.e., by substituting equations (5) and (6) in equation (3) and setting

\[
\delta \tilde{L}/\delta a_k = 0, \quad \delta \tilde{L}/\delta b_k = 0, \quad k = 1, \ldots, N.
\]

These \( 2N \) simultaneous linear complex algebraic equations involve \( \omega^2 (= \lambda) \), \( a_k \) and \( b_k \) as unknowns, and can be expressed in the form of a standard eigenvalue problem,

\[
[A - \lambda B]\{X\} = 0,
\]

where \( \{X\} = [a_1, b_1, a_2, b_2, \ldots, a_k, b_k, \ldots, a_N, b_N]^T \). Solution of the eigenvalue problem (8) gives the complex eigenvalues and eigenvectors. The square root of the real part of the eigenvalue gives the natural frequency, while the ratio of the imaginary to the real part gives the system modal loss factor.

Let us consider some simple cases without damping. For simply supported boundary conditions the functions assumed are

\[
w^k(x) = \sin(k\pi x/L), \quad \alpha^k(x) = \cos(k\pi x/L).
\]

These functions satisfy the geometric boundary conditions and, for an orthotropic or isotropic material beam, they represent the exact mode shapes. Also, since these functions are orthogonal, the equations obtained for a particular value of \( k \) involve coefficients of \( a_k \) and \( b_k \) only. Thus, for each value of \( k \), \( N \) pairs of equations which are uncoupled from each other are obtained (but the two equations in each pair are coupled). Also, for every pair of equations, the general closed form solution is obtained as

\[
\lambda_k = \omega_k^2 = \{k^2\pi^2/L^2(GKRI + TIEI) + TIGK + [k^4\pi^4/L^4(GKRI - TIEI)^2 + (TIGK)^2 + 2k^2\pi^2/L^2TIGK(GKRI + TIEI)]^{1/2}\}/(2RITI).
\]
The Euler solution for a simply supported beam is
\[ \lambda_k = k^2 \pi^2 / L^2 \sqrt{(EI / TI)}. \]  
\[ \text{(11)} \]

For a beam clamped at both ends, the functions used are
\[ w(x) = 1 - \cos (2k\pi x / L), \quad \alpha(x) = \sin (2k\pi x / L). \]  
\[ \text{(12)} \]

These functions satisfy the geometric boundary conditions and are found to give only odd frequencies accurately corresponding to first, third, fifth, etc. modes. Odd modes for a fixed beam are symmetric modes which are predicted here because all functions in the series are symmetric with respect to \( x = L / 2 \). Also, these functions, not being orthogonal, do not uncouple the system of \( 2N \) equations (7). Thus the equations are solved as such in coupled form (8) to give eigenvalues and eigenvectors. For example, for an isotropic Euler–Bernoulli beam, if two terms are taken in the solution, the two frequencies are given by the solution of the quadratic equation
\[ 5\lambda^2 TI^2 - 3\lambda TIEI(2\pi / L)^4 + (4\pi / L)^4 + EI^2(2\pi / L)^4(4\pi / L)^4 = 0. \]  
\[ \text{(13)} \]

This equation gives two frequencies,
\[ \lambda_1 = (r_1 / L)^4 (EI / TI), \quad \text{where } r_1 = 4.741, \quad r_2 = 11.12. \]  
\[ \text{(14)} \]

The actual values from the solution of the beam equation are \( r_1 = 4.731 \) and \( r_2 = 10.997 \). With more terms the convergence towards the odd modal frequencies is improved.

3. SHELL THEORY

3.1. FORMULATION

The Rayleigh–Ritz method has been used for the free vibration analysis. The displacement field is expressed in cylindrical co-ordinates. The mid-plane displacements \( u, v \) and \( w \) in the \( x, \theta \) and \( z \) co-ordinate directions (see Figure 2(b)) are expressed as double series summation functions of \( x \) and \( \theta \). These functions satisfy the geometric boundary conditions. The displacement at any other point is expressed in terms of the mid-plane displacement and the respective rotations \( \psi \) and \( \phi \), which are also expressed as series summation functions:
\[ \bar{u} = u + z\psi, \quad \bar{v} = v + z\phi, \quad \bar{w} = w. \]  
\[ \text{(15)} \]

The magnitude of the strain energy in any stressed state is given by the expression
\[ U = \sum_{j=1}^{n} U_j = \sum_{i=1}^{n} \int_{\gamma} [\sigma_j]^{\top}[e_j] \, dv = \sum_{i=1}^{n} \int_{\gamma} [e_j]^{\top}[Q_j][e_j] \, dv, \]  
\[ \text{(16)} \]
or
\[ \bar{U} = \frac{1}{2} \sum_{i=1}^{n} \int_{\gamma} (Q_{11}\epsilon_{xx}^2 + Q_{22}\epsilon_{zz}^2 + Q_{12}\epsilon_{xx}\epsilon_{zz} + Q_{16}\gamma_{x\theta}^2 + Q_{44}\psi^2 + Q_{55}\gamma_{zz}^2 + Q_{55}\gamma_{xx}^2 + Q_{45}\gamma_{x\theta} + 2Q_{15}\gamma_{x\theta} \gamma_{zz} + 2Q_{15}\gamma_{x\theta} \gamma_{x\theta} + 2Q_{15}\epsilon_{x\theta} \gamma_{zz})(R + z) \, dz \, d\theta \, dx. \]  
\[ \text{(17)} \]

The strain–displacement relations are
\[ \epsilon_x = \bar{u}_x, \quad \epsilon_\theta = (\bar{w} + \bar{v}_x)/(R + z), \quad \gamma_{x\theta} = \bar{\theta}_x + \bar{u}_\theta/(R + z), \]  
\[ \gamma_{xx} = \psi + \bar{w}_x, \quad \gamma_{zz} = (\bar{w}_\theta - \bar{v})/(R + z) + \phi. \]  
\[ \text{(18)} \]
The normal strain in the thickness direction \( \epsilon_z \) has been ignored in these relations. Substituting equations (15) in equations (18) yields the strains in terms of the mid-plane displacements. These strain expressions are substituted in equation (17) and the integration is performed with respect to the \( z \) co-ordinate. The summation for all the layers is also performed at this stage, which gives the total strain energy expression as

\[
\tilde{U} = \frac{1}{2} \int_0^L \int_0^{2\pi} [(Ra_{11} + B_{11})u_x^2 + 2(RB_{11} + D_{11})u_x\psi_x + (RD_{11} + F_{11})\psi_x^2 \\
+ \alpha_{22}(w^2 + v_z^2 + 2uv) + (B_{22} - RA_{22} + R^2\alpha_{22})\phi_y^2 + 2(A_{22} - Ra_{22})(w\phi_y + \phi_y v) \\
+ (Ra_{66} + B_{66})w_x^2 + 2(RB_{66} + D_{66})w_x\phi_x + (RD_{66} + F_{66})\phi_x^2 + 2A_{66}\psi_x u_x \\
+ 2B_{66}(u_x\psi_y + \phi_x) + 2D_{66}\psi_y \phi_x + (A_{44} - Ra_{44})(R^2\phi_x^2 + 2Rw\phi_y - 2Rv\phi_x \\
+ \alpha_{66} u_x^2 + 2(A_{66} - Ra_{66})w_x\psi_y + (B_{66} - RA_{66} + R^2\alpha_{66})\psi_y^2 \\
+ \alpha_{22} \psi^2 - 2vw) + \alpha_{66} w_x^2 + 2A_{12}(u_x w + u_x v) \\
+ 2B_{12}(u_x \phi_y + \psi_x v + \psi_x \phi_x) + 2D_{12}\psi_x \phi_y + 2(RA_{16} + B_{16})u_x v_x \\
+ 2(RB_{16} + D_{16})(\psi_x v_x + u_x \phi_x) + 2(RD_{16} + F_{16})\psi_x \phi_x + 2A_{16} u_x u_x \\
+ 2B_{16}(u_x \psi_y + \psi_x v + \psi_x \phi_x) + 2D_{16}\psi_x \phi_y + 2A_{26}(w_w + u_x v) \\
+ 2B_{26}(w\phi_x + \phi_x v + v_x \phi_y) + 2D_{26}\phi_x \phi_y + 2\alpha_{26}(w w_x + u_x v) \\
+ 2(A_{26} - Ra_{26})(w\phi_y + v_x \phi_y + \phi_y \phi_y) + 2(B_{26} - RA_{26} + R^2\alpha_{26})\phi_y \psi_y \\
+ 2\alpha_{45}(R\phi_x + \psi w - \psi v + R\phi w_x + \psi x v_x - w v_x)] d\phi d\theta. \tag{19}
\]

The thickness shear terms have been multiplied by a correction factor \( k^2 = \pi^2/12 \) in order to compensate for the distribution of the thickness shear strain through the wall thickness. The value of the correction factor used has been derived in Mirsky's formulation for thick shells [5] and was used by Vanderpool and Bert [6] for monoclinic thick cylinders. Also, in the energy expression (19),

\[
[A_{ij}, B_{ij}, D_{ij}, F_{ij}, \alpha_{ij}] = \sum_{i=1}^{n} Q_{ij} \int_{h_{i-1}}^{h_i} [1, z, z^2, z^3, 1/(z + R)] dz. \tag{20}
\]

It may be noted that the \( Q_{ij} \) are functions of the various moduli. To account for the material damping, the moduli are taken in complex form, with the real and imaginary parts representing the storage modulus and the loss modulus, respectively. Thus the strain energy \( U \) obtained from equation (19) is also in complex form. For an orthotropic configuration, the terms with subscripts 16, 26 and 45 will vanish. For symmetric orientations, bending/stretching coupling terms \( B_{ij} \) and higher order terms \( F_{ij} \) will be zero. Terms with subscripts 44 and 55 represent thickness shear contributions to the strain energy in the \( \theta-z \) and \( x-z \) planes, respectively. For an \( n \)-laminated laminate, the layers are numbered in increasing order from the outer to the inner radius.

The kinetic energy is expressed in terms of the displacement components as

\[
\tilde{T} = \frac{1}{2} \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} \omega^2 (\ddot{u}^2 + \ddot{v}^2 + \ddot{w}^2) \rho (R + z) dz d\phi dx. \tag{21}
\]
Substituting equations (15) in equation (21) and performing the integration through the thickness co-ordinate yields the kinetic energy in terms of the mid-plane displacements and the respective rotations as

$$\tilde{T} = \frac{1}{2} \rho \omega^2 \int_0^L \int_0^{2\pi} \left[ R_t (u^2 + v^2 + w^2) + (u\psi + v\phi) (\tau^3/6) + (R_t^3/12) (\psi^2 + \phi^2) \right] \, d\theta \, dx. \quad (22)$$

In the absence of external work the Lagrangian is

$$\mathcal{L} = \mathcal{O} - \tilde{T}. \quad (23)$$

It may be noted that since \( \mathcal{O} \) is in complex form, the Lagrangian \( \mathcal{L} \) will also be complex.

3.2. SOLUTION EQUATIONS

The solution equations are obtained by setting the derivatives of \( \mathcal{L} \) in equation (23) with respect to the solution coefficients equal to zero. The displacement functions are assumed for \( u, v, w, \phi \) and \( \psi \) in double series summation form as

$$u = \sum_{j=1}^{M} \sum_{k=1}^{N} U_{jk} f_j^u(x) g_k^u(\theta), \quad v = \sum_{j=1}^{M} \sum_{k=1}^{N} V_{jk} f_j^v(x) g_k^v(\theta), \quad w = \sum_{j=1}^{M} \sum_{k=1}^{N} W_{jk} f_j^w(x) g_k^w(\theta),$$

$$\phi = \sum_{j=1}^{M} \sum_{k=1}^{N} \Phi_{jk} f_j^\phi(x) g_k^\phi(\theta), \quad \psi = \sum_{j=1}^{M} \sum_{k=1}^{N} \Psi_{jk} f_j^\psi(x) g_k^\psi(\theta). \quad (24)$$

Substituting equations (24) in equation (23), performing the integrations with respect to \( x \) and \( \theta \) and setting

$$\partial L/\partial U_{jk} = 0, \quad \partial L/\partial V_{jk} = 0, \quad \partial L/\partial W_{jk} = 0, \quad \partial L/\partial \Phi_{jk} = 0, \quad \partial L/\partial \Psi_{jk} = 0,$$

$$j = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, N, \quad (25)$$

yields \( 5MN \) complex simultaneous coupled algebraic equations in terms of \( \omega^2 = \lambda \), \( U_{jk}, V_{jk}, W_{jk}, \Phi_{jk}, \Psi_{jk} \) as unknowns. These equations can be expressed in the form of a standard eigenvalue problem,

$$[A - \lambda B] \{X\} = 0, \quad (26)$$

where \( \{X\} = [U_{11}, U_{12}, \ldots, U_{1N}, U_{21}, \ldots, U_{MN}, V_{11}, \ldots, V_{MN}, W_{11}, W_{12}, \ldots, W_{MN}, \Phi_{11}, \Phi_{12}, \ldots, \Phi_{MN}, \Psi_{11}, \Psi_{12}, \ldots, \Psi_{MN}]^T \). \( \{X\} \) is a vector of order \( 5MN \). \( A \) and \( B \) are square symmetric matrices of order \( 5MN \), and their elements are evolved by differentiating \( \mathcal{O} \) and \( \tilde{T} \), respectively. Thus the matrix \( A \) has complex terms while \( B \) is real. Equation (26) is solved for the eigenvalues and corresponding eigenvectors. As explained in the case of beam analysis, the natural frequency is obtained from the real part of the complex eigenvalue, while the system modal loss factor is given by the ratio of the imaginary to the real part of the complex eigenvalue. The eigenvectors are substituted in the displacement functions (24) to obtain the mode shapes. From the mode shapes, the frequencies corresponding to the various modes can be identified.

For a simply supported shell, the functions assumed in the series solution for the displacements are

$$f_j^u(x) = f_j^v(x) = \cos(j\pi x/L), \quad f_j^w(x) = f_j^\phi(x) = \sin(j\pi x/L),$$

$$g_k^u(\theta) = g_k^v(\theta) = g_k^\phi(\theta) = \cos(k\theta), \quad g_k^w(\theta) = g_k^\psi(\theta) = \sin(k\theta). \quad (27)$$
For a clamped shell, the trigonometric functions satisfying the geometric boundary conditions similar to those used in beam theory are assumed, which give the frequencies and loss factors corresponding to the odd numbered modes only:

\[
f_i^*(x) = f_i^*(x) = \sin \left( \frac{2\pi x}{L} \right), \quad f_i^*(x) = f_i^*(x) = 1 - \cos \left( \frac{2\pi x}{L} \right),
\]

\[
g_i^*(\theta) = g_i^*(\theta) = g_i^*(\theta) = \cos (k\theta), \quad g_i^*(\theta) = g_i^*(\theta) = \sin (k\theta).
\]

(28)

4. RESULTS AND DISCUSSION

The various modes of composite shells have been extensively analyzed in the literature by using different theories, which differ mainly in the level of approximation used in the geometric configuration (in terms of powers of \(t/R\), which are retained), or in the distribution of the thickness shear across the wall thickness. In the analysis by Vanderpool and Bert [6] and by Bert and Kumar [14], the numerical results are given for an isotropic shell and a boron/epoxy crossply orthotropic shell, respectively, as obtained by using different theories. Natural frequencies obtained by the present method compared with the reported results are given in Table 1. The results reported in the table, apart from confirming the validity of the programs developed, indicate that for the shell configurations considered the equivalent modulus beam theory gives quite accurate values of the flexural frequencies.

The effects of various parameters on the frequency and system loss factors were studied for a glass epoxy tube under simply supported end conditions. The various moduli \(E_{11}, E_{22}, G_{12}, G_{23}, G_{13}\) and \(v_{12}\) were taken in complex form. The loss factor, which is the ratio of the imaginary to real part of the modulus, was assumed to be zero for \(E_{11}\) and \(v_{12}\), while for \(E_{22}, G_{12}, G_{23}\) and \(G_{13}\) it was assumed to be the same for all and equal to 0.1. The mean radius of the shell was taken as \(R = 0.05\) m. The following values were assumed for the real components of the moduli: \(E_{11} = 70\) GPa, \(E_{22} = 10\) GPa, \(G_{12} = G_{23} = G_{13} = 7\) GPa and \(v_{12} = 0.25\).

**Table 1**

A comparison with reported results for (I) an isotropic shell [6] \((E = 203\) GPa, \(v = 0.285, \rho = 7840\) kg/m\(^3\), \(L = 0.298\) m, \(R = 0.148\) m, \(t = 0.51\) mm) and (II) an orthotropic shell [14] \((L = 0.8\) m, \(R = 0.063\) m, \(t = 0.51\) mm, \(E_{11} = 214\) GPa, \(E_{22} = 18.6\) GPa, \(v_{12} = 0.28, G_{12} = 5.17\) GPa)

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<td>Thin shell</td>
<td>ST2</td>
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<tr>
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<td>532</td>
<td>533-5</td>
<td>533-4</td>
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<td>236-4</td>
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<tr>
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<td>1287</td>
<td>1289</td>
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<td>1290</td>
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<tr>
<td>2 (2)</td>
<td>682</td>
<td>683</td>
<td>675</td>
<td>680</td>
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The actual values of the moduli and loss factors in specific cases may be different. The objective of the present analysis, however, is to make a qualitative study of the effects of various parameters on the natural frequencies and system damping.

4.1. VARIATION OF THE NATURAL FREQUENCIES WITH $L/R$ AND $t/R$

The natural frequencies calculated by using the beam and shell theories are plotted with respect to the tube parameter $L/R$ for a given value of $t/R$. Figures 3(a) and 3(b) are for $t/R = 1/166$: i.e., for a very thin shell with $\theta_s = 0^\circ$. The general observation from these graphs is that the flexural frequency values calculated by the Timoshenko beam theory (TBT) match well with those obtained from the shell theory with thickness shear deformation (ST2).

From Figure 3(a), for the first flexural frequency, we note that for $L/R$ approximately from 6 to 600, TBT and ST2 results match very well. The Euler–Bernoulli beam theory (EBT) without thickness shear gives close results only for $L/R > 60$. The difference between the shell and beam theory values reduces as $L/R$ is increased. This is primarily due to the distortion of the cross-section being greater at lower values of $L/R$. The variation of the second flexural frequency as computed by different theories is shown in Figure 3(b). It is seen that ST2 and TBT give identical results for $L/R$ from about 12 to 2000. The increase in the lower as well as the upper limit, as compared with Figure 3(a), is due to the fact that in the second flexural mode the same type of deformation occurs in each of the two half-lengths of the beam. In other words, a simply supported beam of a given length vibrating in its second flexural mode can be considered equivalent to a half-length of the beam vibrating in its first mode. Thus the factors which cause the difference between the beam and shell theories now become negligible at double the values of $L/R$. This argument would, however, be strictly valid only for simply supported boundary conditions.

Also, in the present shell theory, if thickness shear terms are neglected (ST1), the frequency results predicted still remain remarkably close to those of Timoshenko beam theory but not those of the Euler–Bernoulli beam theory. This can be explained by the
fact that in shell theory the effect of the in-plane shear is of the same nature as the thickness shear effect in beam theory. A comparison of the two shear phenomena, as shown in Figures 4(a) and 4(b), establishes this similarity. For extreme values of \( L/R \), e.g., for \( L/R > 200 \), shear deformable shell theory results still remain close to beam theory values, but ST1 values depart from these.

The effect of changing the thickness is shown in Figure 5. The ratio (\( \beta \)) of the natural frequency calculated from shell theory (ST2) to the Timoshenko beam theory value is plotted against \( L/R \) for different \( t/R \) ratios for \( \theta_p = 0^\circ \). Thus, these curves give a direct comparison between results obtained from the two theories, and indicate limits on \( L/R \) for which these results match. The range of \( L/R \), common for all \( t/R \) ratios considered, for which the first flexural frequency values from the two theories are almost the same (\( \beta = 1 \)) is approximately \( L/R = 6-600 \). Similarly, for \( L/R \) from about 15 to 2000, the second flexural frequency values given by the two theories match. For a larger \( t/R \) ratio, the lower limit on \( L/R \) is lowered further. This is obvious, because the difference occurs due to cross-sectional deformation which would be less for thicker shells.

In Figure 6(a) is shown the variation of the flexural frequencies for extremely low, sufficiently large and very large values of \( L/R \), i.e., 2, 20 and 200 respectively, with respect to \( t/R \) calculated from ST2 theory for \( \theta_p = 0^\circ \). For each \( L/R \) case, the frequency values have been normalized with respect to the flexural frequency obtained for the lowest \( t/R \) (=0-012). For a low value of \( L/R = 2 \), the flexural frequencies increase quite significantly.

Figure 4. The nature of shear deformation: (a) thickness shear in beam theory; (b) in-plane shear in shell theory; (c) thickness shear in shell theory.
Figure 5. Variation of frequency ratio (ST2/TBT) with $L/R$ for different values of $t/R$: (a) 11 mode, (b) 21 mode. $t/R = 1/166.7$, $t/R = 1/16.67$, $t/R = 1/3.33$.

with increasing $t/R$. This thickness-dependent effect could be due to the participation of cross-sectional deformation which, being greater at smaller $t/R$, reduces the frequency. At larger values of $L/R$, however, the deformation of the cross-section is negligible, and thus the frequency remains fairly constant. For example, for $L/R = 20$, if $t/R$ is varied from $1/80$ to $1/3.3$ the increase in the first flexural frequency is approximately 1%, and that in the second flexural frequency is about 2%.

Figure 6. The effect of $t/R$, for different $L/R$ ratios, on (a) flexural frequencies and (b) system loss factors, for $\theta_v = 0^\circ$, according to ST2 analysis.
This marginal increase is also predicted if we use beam theory. The reason for this slight increase is that the moment of inertia also increases with the thickness. However, as can be seen from the expression for $EI$ given following equation (3), the increase in the moment of inertia is of a higher order, being proportional to $t^2/R^2$. Thus, the resultant increase in frequency is significantly smaller.

It is observed that the ratio of the second to the first flexural frequency, which should be 4 in the absence of shear deformation, is approximately 3.99 for $L/R = 200$, 3.2 for $L/R = 20$ and 1.2 for $L/R = 2$. This can be explained on the basis of Timoshenko shear deformation.

For the explanation of the above trends, we have made use of the fact that distortion of the cross-section as predicted by shell theories is greater at smaller $L/R$. This effect is obvious, because for small $L/R$ ratios we expect the tube to behave more like a shell and less like a beam. The extent of the distortion can be predicted from the mode shapes obtained by substitution of the eigenvectors (solution coefficients) in the displacement field. In Figure 7 is shown the deformed cross-section at the point of maximum flexure in the first flexural mode at two values of $L/R$. The resultant displacement at any point on the shell is the vector sum of the displacement components $u$, $v$ and $w$. At any location, $u$, $v$ and $w$ can be written in the form

$$w = a \cos \theta, \quad v = -b \sin \theta, \quad u = c \cos \theta,$$

where $a$, $b$ and $c$ are positive quantities. The resultant of $v$ and $w$ at every point along the circumference indicates the extent of the distortion which the cross-section has undergone. It can be easily shown that for no distortion $b/a = 1$. The lesser the ratio $b/a$, the greater is the distortion of the cross-section.

4.2. VARIATION OF THE NATURAL FREQUENCY WITH THE PLY ANGLE

The first two flexural frequencies are plotted with respect to the ply angle for two values of $L/R = 2$ and 20, for $t/R = 1/3.33$ and $t/R = 1/16.67$ in Figures 8(a)–(d). The observations, with possible explanations, are as follows.

In Figure 8(a), the first and second flexural frequencies have maxima at about $30^\circ$ and $45^\circ$, respectively, when calculated by TBT and at $50^\circ$ and $65^\circ$ when calculated by ST2
theory. Such a high value of the ply angles at which the maxima occur reveals that a large amount of shear deformation is taking place. For $L/R = 20$, in Figure 8(c), the angle corresponding to the maximum frequency shifts towards $0^\circ$, because of the reduction in the effect of shear deformation. A similar trend is observed for thicker shells, as shown in Figures 8(b) and 8(d) for $t/R = 1/3.33$.

The fact that the maximum in the flexural frequency plot occurs due to shear deformation effects can be explained as follows. We know that shear deformation in a beam reduces its frequency, and the extent of the reduction is greater if the ratio of the longitudinal to the shear modulus $(E/G)$ is greater. The shear modulus $G_{xy}$ varies with ply angle, being a minimum at $0^\circ$ and $90^\circ$ and a maximum at $45^\circ$. Similarly, $E_{xy}$ also varies with ply angle, being a maximum at $0^\circ$ and a minimum at $90^\circ$. Thus, as the ply angle is changed from $0^\circ$ to $90^\circ$ the decrease in the longitudinal modulus $E_{xx}$ tends to reduce the frequency. Also, the shear modulus increases from $0^\circ$ to $45^\circ$, leading to a reduced shear deformation effect, which in turn tends to increase the natural frequency. The latter effect dominates, resulting in a maximum frequency not at the $0^\circ$ ply angle but at a higher value. The extent of the shear deformation effect present can be judged by taking the ratio of the second to the first flexural frequency. It may be noted that from beam theory (TBT), for $L/R = 20$ and $t/R = 1/16.67$, this ratio is $3.16$ for $0^\circ$, $3.72$ for $30^\circ$, $3.85$ for $45^\circ$, and $3.76$ for $90^\circ$.

![Figure 8. The flexural frequency vs. the ply angle for: (a) $L/R = 2$, $t/R = 1/16.67$; (b) $L/R = 2$, $t/R = 1/3.33$; (c) $L/R = 20$, $t/R = 1/16.67$; (d) $L/R = 20$, $t/R = 1/3.33$. ———, first flexural mode TBT value; ———, first flexural mode ST2 value; ———, second flexural mode TBT value; ———, second flexural mode ST2 value.](image-url)
Figure 9. The system loss factor vs. $L/R$ for $\theta_p = 0^\circ$ using TBT and ST2: (a) $t/R = 1/3.33$; (b) $t/R = 1/16.67$. First flexural mode TBT value; $\cdots \cdots$, first flexural mode ST2 value; $\cdots \cdots$, second flexural mode TBT value; $\cdots \cdots$, second flexural mode ST2 value.

The frequency maximum as predicted by shell theory is at a still higher angle. In the calculation by shell theory the additional phenomenon of deformation of the cross-section is considered. It has been observed in Figures 4 and 5 that the cross-sectional distortion reduces the natural frequency. Thus this effect acts in conjunction with the beam-type thickness shear deformation. The distortion is mainly dependent on the circumferential modulus $E_{th}$, and reduces as ply angle is increased from $0^\circ$ to $90^\circ$. Second, according to this effect, the difference between the TBT and ST2 values should be minimum at $90^\circ$, which is generally observed in all the Figures 8(a)-(d).

Another interesting phenomenon which governs the difference between the TBT and ST2 values with respect to fibre angle is that the beam-type thickness shear involves both $G_{th}$ and $G_{tr}$. However, in beam theory we consider only $G_{th}$ as the shear modulus. The shell theory actually separates out the effects due to $G_{th}$ and $G_{tr}$. At $\theta = 0^\circ$ and $90^\circ$, $G_{th} = G_{tr}$. However, as $\theta$ increases $G_{th}$ changes (increases up to $45^\circ$ and then drops down), whereas $G_{tr}$ remains the same. Therefore, according to this phenomenon, the difference in the results obtained by the two theories should be a maximum at $45^\circ$ and a minimum at $0^\circ$ and $90^\circ$.

By comparing Figures 8(a) and 8(b), it can be seen that $t/R$ also affects the variation of the frequencies with respect to the ply angle, although the effect on the fundamental frequency is not very significant and is negligible at larger values of $L/R$ (see Figures 8(c) and (d)). The main difference between Figures 8(a) and 8(b) is that at smaller values of the fibre angle the frequencies corresponding to the second flexural mode calculated by the beam and shell theories are far apart for low $t/R$ values, whereas for larger $t/R$ these frequency values almost converge at $\theta_p = 0^\circ$. This result supports the arguments given above. The cross-sectional deformation, which will be greater for smaller $t/R$ and small $\theta_p$, will have a tendency to separate shell theory results from the predictions of beam theory.

4.3. EFFECTS OF $L/R$ AND $t/R$ ON THE SYSTEM LOSS FACTOR

The system loss factor calculated from the beam and shell theories is plotted with respect to $L/R$ for two values of $t/R$ in Figures 9(a) and 9(b) (the variation with $t/R$ for various $L/R$ ratios has already been shown in Figure 6(b)), and its variation with $\theta_p$ for various $L/R$ and $t/R$ is shown in Figures 10(a) and 10(b) of section 4.4. The general conclusion
from surveying all these figures is that the trends of the variation of the system loss factor change sharply with a change in any one of the parameters, as compared to the corresponding variations for natural frequencies. In other words, the loss factor is more sensitive to parametric changes than the frequency.

Figures 6 and 9 are drawn for unidirectional fibres laid parallel to the tube axis: i.e., $\theta_\alpha = 0^\circ$. The loss factor corresponding to $E_{xx}$ (which is the same as $E_{11}$ in this case) is zero, while that corresponding to $G_{\alpha\theta}$ (which is the same as $G_{12}$ in this case) is 0·1. This means that no energy is dissipated in pure longitudinal bending. In beam theory deformations, the only mechanism of energy dissipation is through transverse shear. It is due to this reason that in Figures 9(a) and 9(b) the loss factor predicted by TBT decreases monotonically with an increase in $L/R$, because of the diminishing contribution of shear deformation. However, the variation of the loss factor predicted by ST2 theory is different, and the difference can be explained as follows.

Shell theory takes into account the cross-sectional deformation which provides an additional mechanism for energy dissipation. The effects of cross-sectional deformation are more pronounced at lower $L/R$ and $t/R$ ratios. Now, when the tube bends, and the bending is accompanied by distortion of the cross-section, then different sections along the length of the tube distort to different extents. This means that the resistance to distortion is two-fold: i.e., due to the circumferential modulus $E_{\theta\theta}$ and also due to the longitudinal modulus $E_{xx}$. For low $L/R$ values, the resistance due to $E_{xx}$ is predominant, corresponding to which there is no dissipation. This explains the decreasing of the loss factor curves in the low $L/R$ region in Figures 9(a) and 9(b). However, as one increases $L/R$, the relative contribution to the strain energy from $E_{\theta\theta}$ increases, and as its loss factor is 0·1 this raises the system loss factor. Thus, beyond a particular limit of $L/R$, the loss factor predicted by ST2 becomes greater than the TBT value, as shown in Figures 9(a) and 9(b).

Also as $L/R$ is increased further, the distortion itself continues to decrease, which is reflected in the observation that the difference between the TBT and ST2 values continues to diminish and, for large $L/R$ values, this difference is negligible.

In Figure 6(b) the loss factors have been evaluated by using shell theory with transverse shear deformation (ST2). It may be noted that Figure 6(b) is for a $0^\circ$ fibre angle, with the loss factor of $E_{\theta\theta}$ being zero. In the absence of shear deformation and distortion of the cross-section, the value of the loss factor corresponding to flexural modes should have been nearly zero. However, for $L/R = 2$ (see Figure 6(b)), it is noticed that the loss factors vary from 0·1 and even the minimum value obtained for the 21 mode is 0·055. For the 11 mode the loss factor varies from 0·08 to 0·1. Such high values of the system loss factors reflect the large amount of shear deformation and cross-sectional distortion that is taking place. The cross-sectional deformation involves $E_{\theta\theta}$ which has a loss factor of 0·1. Thus the greater the distortion the greater will be the system loss factor. For $L/R = 20$ the loss factors for the two modes are 0·01947 and 0·04919. Beam-type thickness shear deformation is still contributing to the strain energy and providing its damping capacity. For $L/R = 200$ the loss factors are of the order of 0·00025 and 0·00099 for the first and second flexural modes, respectively. These negligibly small values indicate the near absence of shear deformations.

For $L/R = 2$, the loss factor in general decreases with increasing $t/R$. As $t/R$ increases, more and more beam-type bending replaces the cross-sectional deformation. However, if the thickness is further increased beyond a limit, dissipation may start to increase due to participation of the wall thickness shear. This interesting observation is reflected in the fact that, for $L/R = 2$, the 21 mode loss factor has a minimum at $t/R \approx 1/6$. It may be noted, however, that the above-mentioned trend is true for $0^\circ$ fibre angle only, and will change with a change in ply angle. Comparing the loss factors for flexural modes at $L/R = 20$ and
$L/R = 200$, one finds that the loss factor remains almost constant. This is obviously due to the fact that the deformation of the cross-section is considerably smaller.

4.4. EFFECT OF THE PLY ANGLE ON THE SYSTEM LOSS FACTOR

The variation of the system loss factors corresponding to the various flexural modes with ply angle is shown in Figures 10(a) and 10(b). Comparison of Figures 10(a) and 10(b), which are for two different values of $L/R$, shows that this variation is strongly dependent on the $L/R$ ratio. In Figure 10(a), for $L/R = 2$, we observe that the relative magnitudes of loss factor calculated by the two theories change with ply angle. First, consider the curves for the TBT results. These loss factors show a minima: e.g., the 11 mode loss factor is a minimum at about 30$^\circ$ and the 21 mode loss factor a minimum at 40$^\circ$. Similar to the explanation given for occurrence of frequency maxima in Figures 8(a)–(d), the loss factor minima occurs due to two opposing influences on the loss factor: (a) a decrease in the loss factor is caused by a decreasing contribution of the shear deformation which happens due to lowering of the $E/G$ ratio; moreover, the loss tangent of the shear modulus $G_{xy}$ itself continues to reduce as the ply angle is increased. (b) An increase in the loss factor occurs due to an increase in the loss tangent of the longitudinal modulus $E_{xx}$. The dominant influence of the first factor causes a minimum in the loss factor variation. Also, more shear in the 21 mode results in its minimum occurring at a larger ply angle. However, the angles corresponding to the minimum loss factors are much less for $L/R = 20$: viz., 15$^\circ$ for the 11 mode and 20$^\circ$ for the 21 mode (see Figure 10(b)), which is obviously due to less shear deformation.

If we now consider the loss factors predicted by the shell theory, two additional phenomena that occur are distortion of the cross-section and the shear deformation of the shell wall. The effect of the distortion itself is complex and occurs in more than one way. The additional factors which influence the trends of variation of loss factor are as follows. (c) At very low angles, quite a substantial part of the distortion is resisted by the longitudinal modulus, which corresponds to no dissipation. (d) With an increase in ply angle, $E_{yy}$ increases and its loss modulus reduces, whereas $E_{xx}$ reduces and its loss factor increases in a manner exactly opposite to that of $E_{yy}$. (e) The cross-sectional deformation itself increases or reduces depending on the relative participation of $E_{xx}$ and $E_{yy}$, which

![Figure 10](image_url)

**Figure 10.** The system loss factor $\nu$ vs. the ply angle for $L/R = 1/3.33$, and (a) $L/R = 2$ and (b) $L/R = 20$. ——, first flexural mode TBT value; — — , first flexural mode ST2 value; — — —, second flexural mode TBT value; — — - —, second flexural mode ST2 value.
Table 2
Results for quasi-isotropic shells: the effect of changed sequence—loss factors are indicated in parentheses

<table>
<thead>
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<th>( t/R = 8/150 )</th>
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<td>11</td>
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</tr>
<tr>
<td></td>
<td>21</td>
<td>917.76 (0.02772)</td>
</tr>
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</table>

Further depends on the \( L/R \) ratio. (f) The wall thickness shear provides an additional mechanism of energy dissipation, which has a tendency to increase the system loss factor.

The factors (d) and (e) together could provide possible reasons for the observation that the rate of decrease of the loss factor is more subdued at smaller angles and for the 21 mode it is almost zero up to about 45\(^\circ\), as shown in Figure 10(a). However, for \( L/R = 20 \), in Figure 10(b), this result is not observed because the contribution of the cross-sectional distortion itself reduces.

4.5. Effect of the Stacking Sequence on the Frequency and the Loss Factor

Some results obtained by using shell theory S12, for three eight-layered quasi-isotropic laminated shells with two different \( t/R \) and \( L/R \) ratios, are shown in Table 2. From the table we note that with a change in stacking scheme from I to II there is a slight decrease in the flexural frequencies. It may be noted that shell modal frequencies showed an opposite trend [15], the relative increase in the frequencies being quite large. Moreover, the change is greater for tubes with smaller \( L/R \) and \( t/R \) values. For example, for \( t/R = 3/50 \), the relative decrease in the first flexural frequency is 0.15% for \( L/R = 2 \) and 0.03% for \( L/R = 20 \). For \( t/R = 8/50 \) the corresponding values are 0.92% for \( L/R = 2 \) and 0.27% for \( L/R = 20 \). At lower \( L/R \), the deformation of the cross-section and the shear of the shell wall are significant. The higher frequency values for scheme I are due to increased longitudinal flexural rigidity as compared to that for scheme II, because the 0\(^\circ\) fibres which provide a high value of flexural stiffness are located away from the mid-plane in scheme I. The relative difference being greater at larger \( t/R \) is due to the fact that the changes in flexural stiffness will be greater for thicker shells.

The loss factors for scheme II are slightly higher as compared to those for scheme I, which is also a direct consequence of the same reasons as are responsible for the decrease in frequency. There is no dissipation for plies with \( \theta_p = 0^\circ \), while dissipation is a maximum in plies at 90\(^\circ\). For scheme II the contribution of longitudinally aligned plies (\( \theta_p = 0^\circ \)) decreases, while that of circumferentially oriented laminas increases. This results in a net increase in the loss factor for scheme II. The increase is greater for smaller \( L/R \) and larger \( t/R \) ratios.
DAMPED FLEXURAL VIBRATIONS OF TUBES

It is important to note here that the changes in the flexural frequency values caused by a change in the stacking sequence cannot be predicted by equivalent modulus beam theory. However, in the case of tube configurations used in practice in composite shaft applications, the differences in the flexural frequencies are negligibly small. This fact is employed to justify the use of equivalent modulus beam theory. The shell theory, however, is free from this limitation.

5. CONCLUSIONS

Natural frequencies and system loss factors have been obtained for flexural modes of composite tubes by using beam and shell theories. The analysis shows that the results from beam theory are more inaccurate under the extreme conditions of low $L/R$ and $I/R$ values when (i) bending is accompanied by cross-sectional deformation, which is more pronounced when $E_{20}$ is much smaller than $E_{22}$, and (ii) beam-type shear deformation effects are large, for the case of low shear moduli.

Moreover, the beam theory presently used cannot account for different values of $G_{12}$ and $G_{22}$ as well as the bending/stretching and shear normal couplings. Subject to these limitations, one can find the limits of tube parameters up to which beam theory will predict reasonably correct values of the flexural frequencies and corresponding modal loss factors.

In shell theory results for flexural modes the distribution of shell-type thickness shear deformation, i.e., deformation of the shell wall, gives rise to differences of lesser order. This was seen by finding the flexural frequencies for different values of $G_{13}$ and $G_{23}$.

The present analysis shows that the loss factors are in general more sensitive to parametric changes than the frequencies, but the conditions leading to inaccuracy of the beam theory results are the same as those for the natural frequencies.

REFERENCES


**APPENDIX: NOMENCLATURE**

\[ A_{ij} \] in-plane stiffness of the laminated shell  
\[ B_{ij} \] bending/stretching coupling stiffness  
\[ D_{ij} \] flexural stiffness  
\[ E \] equivalent linear elastic modulus of the laminate in the longitudinal direction  
\[ E_i \] linear elastic modulus in the \( i \)-direction \( (i = 1, 2, x, \theta) \)  
\[ G \] equivalent shear modulus of the laminate  
\[ G_{ij} \] shear modulus in the \( i-j \) plane \( (i, j = 1, 2, x, \theta) \)  
\[ L \] length of the cylindrical tube  
\[ [Q_{ij}] \] stiffness matrix of the \( i \)th ply with respect to the shell axes  
\[ R \] mean radius of the cylindrical tube  
\[ U_1, U_2, U_3, U_4, U_5 \] laminate invariants  
\[ U_i \] strain energy contribution due to the \( i \)th lamina  
\[ x, \theta, z \] cylindrical co-ordinates (also used as subscripts)  
\[ 1, 2, 3 \] material symmetry axes for the lamina (also used as subscripts)  
\[ u, v, w \] displacements in the \( x, \theta \) and \( z \) co-ordinate directions, respectively  
\[ \theta_i \] fibre orientation angle in the \( i \)th ply  
\[ \psi \] rotation in the \( x-z \) plane (about the \( \theta \)-axis)  
\[ \phi \] rotation in the \( \theta-z \) plane (about the \( x \)-axis)  
\[ \{\varepsilon_i\} \] strain tensor at the \( i \)th lamina with respect to the shell axes  
\[ \{\sigma_i\} \] stress tensor at the \( i \)th lamina with respect to the shell axes  
\[ \rho \] density of the laminate material  
\[ \lambda_i \] eigenvalues of the dynamic matrix  
\[ EBT \] Euler–Bernoulli beam theory (without thickness shear)  
\[ TBT \] Timoshenko beam theory  
\[ ST1 \] Shell theory, without the thickness shear deformation terms in the strain energy expression  
\[ ST2 \] shear deformable shell theory