CONVECTIVE HEAT TRANSFER ACROSS A HORIZONTAL IMPERMEABLE PARTITION

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Abstract—The aim of this paper is to study the natural convection flow on both sides of an impermeable horizontal partition separating two semi-infinite fluid saturated porous reservoirs maintained at different temperatures. As a result conjugate boundary layers are formed on both sides of the partition. Heat is transferred from the hot side to the cold side through a thermally conducting partition. It is shown that an increase in thermal resistance of the wall reduces the net heat transfer rate across the partition.

INTRODUCTION

An exhaustive study of natural convection flow in porous media reveals its need in geophysical and energy related engineering problems. A very common problem encountered in various applications is the study of heat exchange between two fluid-saturated porous media separated by a wall. The impermeable wall heats the fluid on one side and is itself heated on the other side giving rise to buoyancy induced circulations on both the sides. Such a phenomenon finds its application in various fields such as building insulation, electronic circuitry cooling, fin heat transfer, energy storage in enclosures, furnace design and convective components of habitable spaces [1]. Inspite of its important applications, there are only a few [2–6] investigations dealing with the growth of conjugate boundary layers across vertical partitions. The purpose of this study is to analyse the corresponding problem of conjugate heat transfer across horizontal partition.

The geometry of the physical problem under study consists of a flat horizontal surface separating two fluid-saturated semi-infinite porous spaces maintained at different temperature due to which boundary layers are formed on both sides of the wall (Fig. 1). However, the wall temperature or heat flux is not known a priori but is determined from the heat transfer interaction between the two natural boundary layers. The unknown condition at the wall presents a difficulty in finding the solution. It is resolved by assuming a profile for the temperature and velocity field satisfying the boundary conditions. The set of equations thus obtained can then be successively solved by employing the Karman–Pohlhausen integral method [7].

PHYSICAL MODEL AND MATHEMATICAL FORMULATION

Invoking Boussinesq approximation and considering Darcy flow model, the governing equations for the physical problem under study for constant physical properties of the porous medium and the fluid are

\[
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0
\]

(1)

\[
u^* = \frac{-K}{\mu} \frac{\partial p^*}{\partial x^*}
\]

(2)

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\[ u^* = -\frac{K}{\mu} \frac{\partial p^*}{\partial y^*} + \frac{Kg\beta}{\nu} \left[ T^* - \frac{1}{2} (T_h^* - T_c^*) \right] \] (3)

\[ u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \alpha \frac{\partial^2 T^*}{\partial y^* \partial x^*}. \] (4)

Elimination of the pressure gradient from equations (2) and (3) gives

\[ \frac{\partial u^*}{\partial y^*} - \frac{\partial v^*}{\partial x^*} = -\frac{Kg\beta}{\nu} \frac{\partial T^*}{\partial x^*}. \] (5)

From a scale analysis it follows that \( o(\partial u^*/\partial x^*) \ll o(\partial u^*/\partial y^*) \). Hence, neglecting the second term in equation (5) we arrive at

\[ \frac{\partial u^*}{\partial y^*} = -\frac{Kg\beta}{\nu} \frac{\partial T^*}{\partial x^*}. \] (6)

Equations (1), (6) and (4) in non-dimensional form can be written as (see Nomenclature)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] (7)

\[ \frac{\partial u}{\partial y} = \frac{\partial T}{\partial x} \] (8)

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2}. \] (9)

The temperature boundary conditions are

\[ T \to -\frac{1}{2} \text{ as } y \to \infty \]

\[ T \to \frac{1}{2} \text{ as } y \to -\infty \] (10)

and at the wall

\[ \frac{\partial T}{\partial y} \bigg|_{y=-w/2} = \frac{\partial T}{\partial y} \bigg|_{y=w/2} \]

\[ T(x, -w/2) - T(x, w/2) = 2Q \] (12)
where $2Q$ is the dimensionless temperature difference between the two faces of the partition; $Q = 0$ implies a partition of negligible wall thickness or equivalently, a partition of infinite thermal conductivity. Equation (11) recognizes the fact that heat flux is conserved as it passes through the wall.

For free convection flow across an impermeable surface the appropriate velocity boundary conditions are

$$u = 0 \quad \text{as} \quad y \to \pm\infty$$
$$v = 0 \quad \text{at} \quad y = \pm w/2.$$  

(13)  
(14)

**SOLUTION PROCEDURE**

The temperature and velocity profiles which satisfy the boundary conditions (10), (12) and (13) are assumed as [3]

$$T_c = \left(T_0 - Q + \frac{1}{2}\right)e^{-\left(y-w/2\right)/\delta_c} - \frac{1}{2}$$

$$T_h = \left(T_0 + Q - \frac{1}{2}\right)e^{\left(y+w/2\right)/\delta_h} + \frac{1}{2}$$

$$u_c = u_{wc}e^{-\left(y-w/2\right)/\delta_c}$$

$$u_h = u_{wh}e^{\left(y+w/2\right)/\delta_h}$$

(15a)  
(15b)  
(16a)  
(16b)

$u_{wc}$, $u_{wh}$ are respectively the $x$-component of the velocity on the cold and warm sides of the wall. They are calculated in terms of the mid-plane temperature $T_0$ and the boundary layers distributions $\delta_c$ and $\delta_h$ from the modified equation of momentum (8). Equation (8) is integrated with respect to $y$ from $w/2$ to $\infty$ and from $-\infty$ to $-w/2$ to yield

$$u_{wc} = \frac{d}{dx}\delta_c\left(T_0 - Q + \frac{1}{2}\right)$$

$$u_{wh} = -\frac{d}{dx}\delta_h\left(T_0 + Q - \frac{1}{2}\right).$$

(17a)  
(17b)

With the help of the equation of continuity (7) and equation of momentum (8), integration of the energy equation (9) with respect to $y$ from $w/2$ to $\infty$ and from $-\infty$ to $-w/2$, results in

$$\frac{d}{dx}\int_{w/2}^{\infty} u_cT_c\,dy + v_cT_c\big|_{w/2}^{\infty} = \left.\frac{\partial T_c}{\partial y}\right|_{w/2}^{\infty}$$

$$\frac{d}{dx}\int_{-\infty}^{-w/2} u_hT_h\,dy + v_hT_h\big|_{-w/2}^{-\infty} = \left.\frac{\partial T_h}{\partial y}\right|_{-\infty}^{-w/2}.\tag{18a}$$

$$\frac{d}{dx}\int_{w/2}^{\infty} u_cT_c\,dy + v_cT_c\big|_{w/2}^{\infty} = \left.\frac{\partial T_c}{\partial y}\right|_{w/2}^{\infty}$$

$$\frac{d}{dx}\int_{-\infty}^{-w/2} u_hT_h\,dy + v_hT_h\big|_{-w/2}^{-\infty} = \left.\frac{\partial T_h}{\partial y}\right|_{-\infty}^{-w/2}.\tag{18b}

Equation (7) is used in order to evaluate the second term in equations (18a, b). The outcome of integrating equation (7) with respect to $y$ with proper limits is

$$v_cT_c\big|_{w/2}^{\infty} = \frac{d}{dx}\int_{w/2}^{\infty} \frac{1}{2}u_c\,dy$$

$$v_hT_h\big|_{-w/2}^{-\infty} = -\frac{d}{dx}\int_{-\infty}^{-w/2} \frac{1}{2}u_h\,dy.\tag{19a}$$

(19b)

On using profiles (15), (16) and equations (17), (19), equations (18a, b) yield a system of coupled differential equations

$$\frac{d}{dx}\left(u_{wc}\left(T_0 - Q + \frac{1}{2}\right)\delta_c\right)^2 = 4u_{wc}\left(T_0 + \frac{1}{2} - Q\right)^2$$

$$\frac{d}{dx}\left(u_{wh}\left(T_0 + Q - \frac{1}{2}\right)\delta_h\right)^2 = 4u_{wh}\left(T_0 - \frac{1}{2} + Q\right)^2.\tag{20a}$$

(20b)
The dimensionless temperature difference $2Q$ across the partition is obtained by equating the conduction heat flux through the wall with the convection flux on either side and by using equation (12). We thus obtain

$$2Q = \frac{\omega \left( T_0 + \frac{1}{2} \right)}{(\delta_c + \omega/2)} \quad (21a)$$

$$2Q = -\frac{\omega \left( T_0 - \frac{1}{2} \right)}{(\delta_h + \omega/2)} \quad (21b)$$

where $\omega/w = k/k_w$ is the ratio of the conductivity of the porous matrix to the conductivity of the wall, $\omega$ being the wall thermal resistance parameter.

Substituting for $u_{we}$ and $u_{wh}$ from equations (17a, b) and for $Q$ from equations (21a, b) into equations (20a, b) and on integrating the resultant equations once with respect to $x$ after using the fact that $z(-1/2) = 0$ and $s(1/2) = 0$ we get the following system of coupled integro-differential equations

$$\left( \frac{dz}{dx} \right)^2 = 8 \left( \frac{1 + T_0}{\delta_c + \omega/2} \right) z - 8 \int_{-1/2}^x \frac{dz}{dx} \left( \frac{1 + T_0}{\delta_c + \omega/2} \right) dx \quad (22a)$$

$$\left( \frac{ds}{dx} \right)^2 = 8 \left( \frac{1 - T_0}{\delta_h + \omega/2} \right) s + 8 \int_x^{1/2} \frac{ds}{dx} \left( \frac{1 - T_0}{\delta_h + \omega/2} \right) dx \quad (22b)$$

where

$$z = \left[ \frac{\delta_c^2 \left( T_0 + \frac{1}{2} \right)^2}{(\delta_c + \omega/2)} \right]$$

and

$$s = \left[ \frac{\delta_h \left( 1 - T_0 \right)^2}{(\delta_h + \omega/2)} \right].$$

Also, from equation (21), $\delta_h$ is expressed in terms of $\delta_c$ and $T_0$ as

$$\delta_h = \frac{\delta_c \left( 1 - T_0 \right) - T_0 \omega}{(\frac{1}{2} + T)}.$$ 

(24)

Equations (22a, b) are solved numerically using an iterative scheme discussed below, with the initial conditions $z(-1/2) = 0$ and $s(1/2) = 0$.

**NUMERICAL SOLUTIONS AND RESULTS**

The following steps were adopted to obtain the solution of the problem under study:

**Step 1.** Appropriate initial profile for $\delta_c$ and $T_0$ are assumed as

$$T_0^{(0)} = x + \frac{1}{4}$$

$$\delta_c^{(0)} = x + \frac{1}{2}.$$  

**Step 2.** Equations (22) are discretized using backward Euler's method to yield $z$ and $s$ at each mesh point at the next iterative level ($k + 1$).
Step 3. The mid-plane temperature \( T_0 \) at \( x = -1/2 \) is obtained from equation (23b) after substituting for \( \delta_n \) from equation (24) and using the condition \( \delta_c(-1/2) = 0 \). This gives
\[
T_0 = \frac{(s)^{1/2} - \left[ s + 4 \omega(s)^{1/2} \right]^{1/2}}{4\omega}.
\] (25)

Step 4. \( T_0 \) and \( \delta_n \) are eliminated from equations (23) and (24) to yield an expression for \( \delta_c \) as
\[
\delta_c = \left( \frac{z^{1/2} + \left(s + 4\omega \right)^{1/4}}{z^{1/2}} \right) + \frac{4\omega z^{1/2}}{2}.
\] (26)

Step 5. \( T_0 \) is evaluated from equation (23a) as
\[
T_0 = \frac{(s)^{1/2}}{\delta_c^2} \left( \delta_c + \frac{\omega}{2} \right) - \frac{1}{2}.
\] (27)

The iterative scheme is carried out until
\[
\text{Max} (err_1, err_2) < \epsilon
\]
where \( err_1, err_2 \) denote the maximum absolute error in \( \delta_c \) and \( T_0 \) respectively and \( \epsilon \) is the tolerance.

Step 6. The tangential components of the velocity on the two sides of the wall are then found by using equations (17).

Figures 2, 3 and 4 illustrate the growth of the boundary layer thickness, mid-plane temperature distribution and the tangential component of the velocity at the wall. In the case of free convective flow across a horizontal partition the buoyancy force \( B_n \) is directed along the normal to the surface. Since the upper surface of the partition is warmer than the ambient temperature at the top, \( B_n \) will be directed along the positive \( y \) axis. The effect of \( B_n \) is to increase the normal component of the velocity \( v \) away from the surface and cause the fluid to tend to lift off the surface. On the lower side of the partition, the surface is cooler than the ambient temperature at the bottom and hence, \( B_n \) is again directed away from the surface and its effect is to increase \( v \) away from the surface in the region below the partition [8]. As there is a similar effect of the buoyancy on both sides of the partition, the boundary layer profile, the mid-plane temperature distribution and the tangential component of the velocity at the wall will exhibit the centrosymmetric property. Furthermore, Fig. 3 shows that the mid-plane temperature decreases with an increase in the wall thermal resistance. This result is expected.
Fig. 3. Variation of the mid-plane temperature $T_0$ with $x$.

Fig. 4. Horizontal component of the velocity $u_x$ vs $x$. 
since an increase in $\omega$ will reduce the thermal contact between the two boundary layers. The tangential component of velocity at the wall also decreases with increase in the wall thermal resistance parameter $\omega$. Moreover, regardless of the value of $\omega$, the tangential velocity $u_w$ is almost constant along the partition (Fig. 4).

The wall flux is presented in Fig. 5 as the vertical temperature gradient $-(\partial T/\partial y)_{x=\omega/2}$. The heat flux decreases with increase in wall thermal resistance $\omega$, that is, with an increase in the effective insulation between the two spaces. It is observed that the heat flux is nearly uniform over most of the length $L^*$ of the partition for higher values of $\omega$.

The values of the average heat transfer rate defined by

$$\overline{Nu/Ra^{1/3}} = \int_{-\omega/2}^{\omega/2} T_0 + \frac{1}{2} \frac{1}{\delta_e + \omega/2} dx,$$  \hspace{1cm} (28)$$

are listed for different values of $\omega$ in Table 1. These values are fairly close to the local heat flux at the centre of the partition. Like in the case of the vertical partition, it is noticed from the

Table 1. Effect of wall thermal resistance parameter $\omega$ on average heat transfer rate

| $\omega$ | $T_0|x=-\omega/2|$ | $\overline{Nu/Ra^{1/3}}$ | $\overline{Nu/Ra^{1/3}}|_{x=0}$ |
|---------|------------------|------------------|------------------|
| 0       | -0.5000          | 0.5221           | 0.4666           |
| 0.5     | -0.3572          | 0.3975           | 0.3651           |
| 1.0     | -0.2901          | 0.3291           | 0.3000           |
| 2.0     | -0.2191          | 0.2327           | 0.2225           |
| 4.0     | -0.1547          | 0.1527           | 0.1482           |
| 6.0     | -0.1299          | 0.1144           | 0.1119           |
| 8.0     | -0.1033          | 0.0918           | 0.0901           |
| 10.0    | -0.0899          | 0.0768           | 0.0756           |
table that an increase in the thermal resistance parameter $\omega$ reduces the net heat transfer rate across the horizontal partition.

CONCLUSIONS

The growth of conjugate boundary layers on the two sides of a horizontal partition embedded in a fluid saturated porous medium and the average heat transferred across it is investigated numerically. The governing equations are subjected to the Karman–Pohlhausen integral approach and the resulting coupled system is solved by an iterative numerical procedure. The velocity and the temperature fields on the two sides of the partition are chosen in a suitable exponential form. The tangential component of the velocity at the wall and the mid-plane temperature are not prescribed a priori but are determined as part of the solution to the problem. The boundary layer and mid-plane temperature distributions are found to exhibit the centrosymmetric property as in the case of vertical partition. The effect of finite wall thermal resistance on the net heat transfer rate has also been examined in this paper. It is shown that the net heat transfer rate reduces with an increase in the thickness of the horizontal partition separating the two ambient fields.

REFERENCES


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NOMENCLATURE

$$g = \text{Acceleration due to gravity}$$

$$h = \frac{-k}{T_h - T_0} \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=0} = \text{Local heat transfer coefficient}$$

$$k = \text{Thermal conductivity of the porous medium}$$

$$K = \text{Permeability of the porous medium}$$

$$L^* = \text{Total length of the horizontal partition}$$

$$Nu = \frac{hL^*}{k} = \text{Nusselt number}$$

$$Q = \text{Dimensionless temperature difference}$$

$$Ra = \frac{Kg\beta L^* (T_h - T_0)}{\alpha} = \text{Rayleigh number}$$

$$T^* = \frac{T - T_h}{T_h - T_0} = \text{Dimensionless temperature}$$

$$L^* = \frac{-Ra}{\alpha} = \text{Dimensionless velocity in x-direction}$$

$$\nu = \frac{L^*}{\alpha} \frac{Ra^{-1/5}u^*}{\nu} = \text{Dimensionless velocity in y-direction}$$

$$w = \frac{W^*}{L^*} \frac{Ra^{-1/5}}{\nu} = \text{Dimensionless wall thickness}$$

Greek letters

$$\alpha = \text{Thermal diffusivity}$$

$$\beta = \text{Coefficient of thermal expansion}$$

$$\delta = \text{Dimensionless boundary layer thickness}$$

$$\nu = \text{Kinematic viscosity of the fluid}$$

$$\omega = \text{Thermal resistance of the wall}$$

Subscripts

$$c = \text{Reference quantity above the partition}$$

$$h = \text{Reference quantity below the partition}$$

$$w = \text{Condition at the wall}$$

$$0 = \text{Condition at the mid-plane}$$

$$n = \text{Ambient condition outside the boundary layer}$$

Superscript

$$^* = \text{Dimensional quantity}$$