STABILITY OF ROTOR SYSTEMS WITH VISCOELASTIC SUPPORTS

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Stability analysis of a rotor system with the rotor disk placed in the middle of a massless shaft, having linear elasticity and internal damping, with linear elastic bearings at the ends and supported on viscoelastic supports, has been carried out. Stability zones have been determined for various parametric values, and results have been compared for cases with the supports being purely elastic or viscously damped and flexible. It is found that suitably chosen viscoelastic supports can increase the stability zones of the system considerably compared to the other types of supports considered.

1. INTRODUCTION

The stability aspect of a rotor system is important since it gives one an idea about the safe speed of rotation of the system. In reference [1], the stability of a single rotor-shaft system has been considered, with the disk placed in the middle of the shaft, supported on rigid supports. Stiffness, internal damping and viscous external damping have been taken into account at the rotor disk station, and it has been shown that the rotor is stable for any rotational speed less than the natural frequency of the system for the case of no external damping. Ehrich [2] carried out an analysis of instability due to rotor internal damping and defined stability boundaries in terms of the ratio of the external system damping to the internal shaft damping. In reference [3], the stability of single mass rotor with internal friction and viscously damped flexible supports was analyzed. It was seen that properly chosen support damping helps in increasing the stability zone and that it decreases with increase in support mass. The stability of journal bearings with a flexible viscously damped support was analyzed [4, 5] and it was shown that the stability threshold can be significantly enhanced compared to that of rigidly mounted bearing by using suitable viscous damping for the case of flexible supports.

The use of viscoelastically damped supports has been considered [6, 7] to be beneficial in reducing rotor resonant response, although investigations on stability analysis of such systems do not appear to have been reported. The same has been attempted in the present work, with a four-element type of model being used to describe the viscoelastic support characteristics, and comparisons have been made for the cases when the supports are purely elastic or viscously damped and flexible.

2. ANALYSIS

The schematic diagram of the arrangement under consideration is shown in Figure 1(a). With the gyroscopic effect and external damping at the rotor disk station ignored, the
equations for free vibrations can be written as follows, upon assuming that the internal damping for the shaft is a linear viscous one, as assumed in references [1–3, 8, 9]:

\[ M_2 \ddot{z}_2 + C_2 \dot{z}_2 + (K_s - i \omega C_i)z_s = 0, \]
\[ K_b z_j - C_b \dot{z}_s - (K_s - i \omega C_i)z_s = 0, \quad M_1 \ddot{z}_1 + K_z z_1 - K_b z_j = 0. \]  

Here \( C_i \) is the internal damping coefficient of the shaft material, \( K_s \) is the stiffness of the shaft at the rotor disk station, and \( K_b \) is the total bearing stiffness. The detailed notations are given in Appendix 2.

As seen from references [10, 11], the four-element viscoelastic model of Figure 1(b) can adequately represent the dynamic properties of some of the common viscoelastic materials and it has been used in the present work. The force-displacement relation for such a model has been given in references [10, 11] and the same can be used to model \( K \), the equivalent support stiffness, so that equation (3) becomes

\[ M_2 \ddot{z}_1 + \{C + (\gamma_1 D + \gamma_2 D^2/(1 + \alpha_1 D))\} z_1 - K_b z_j = 0, \]

where \( \gamma = K_1, \quad \gamma_1 = (K_2/K_s) \eta_2 + \eta_1 + \eta_2, \quad \gamma_2 = \eta_1 \eta_2 / K_2, \quad \alpha_1 = \eta_2 / K_2 \) and \( D \) is the operator \( \frac{d}{dt} \).

Taking \( z_2 = Z_2 e^{i\omega t}, \quad z_j = Z_j e^{i\omega t} \) and \( z_1 = Z_1 e^{i\omega t} \), and with \( z_i = z_2 - z_j - z_1 \), one obtains, from equations (1), (2) and (4),

\[ -M_2 \lambda^2 Z_2 + i C_i \lambda (Z_2 - Z_j - Z_1) + (K_s - i \omega C_i)(Z_2 - Z_j - Z_1) = 0, \]
\[ K_b Z_j - i C_i \lambda (Z_2 - Z_j - Z_1) - (K_s - i \omega C_i)(Z_2 - Z_j - Z_1) = 0, \]
\[ -i \lambda^3 \alpha_1 M_1 Z_1 - \lambda^2 (M_1 + \gamma_2) Z_1 + \gamma_1 i \lambda Z_1 + \gamma Z_1 - i \alpha_1 K_b \lambda Z_j - K_b Z_j = 0. \]

Non-dimensionalizing equations (5), (6) and (7) and rearranging the coefficients yields the equations of motion with non-dimensional coefficients as

\[ \left[ (R^2 - 1) + i (2 \xi R - 2 \xi) \right] Z_2 + \left[ 1 + i (2 \xi R - 2 \xi) \right] Z_j + \left[ 1 + i (2 \xi R - 2 \xi) \right] Z_1 = 0, \]
\[ -1 + i (2 \xi R - 2 \xi) \right] Z_2 + \left[ (1 + \beta) + i (2 \xi R - 2 \xi) \right] Z_j + \left[ 1 + i (2 \xi R - 2 \xi) \right] Z_1 = 0, \]
\[ \left[ \beta \beta_2 + i (2 \xi \beta_2) \right] Z_j + \left[ R^2 (\alpha \beta_2 + 4 \xi \beta_2 - \beta_2) \right] Z_1 + i (2 R^3 \alpha \xi_2 - 2 \xi \beta_2 R - 2 \xi \beta_2 R - 2 \beta_1 \xi_2 R) \right] Z_1 = 0. \]
where $R = \lambda / \omega_n$, $\omega_n$ being the natural frequency of the rotor on rigid supports: i.e., $(K/M_2)^{1/2}$. $\delta = \omega / \omega_n$ is the frequency ratio. The variables $\xi_1, \xi_2, \beta_1, \beta_2$ and $\beta_3$ are as defined in Appendix 2. The damping coefficients have been divided by $C_r = 2M_2\omega_n$, the critical damping of the rotor system on rigid support, to obtain the various damping ratios: i.e., $\xi = C_i/C_r$, called the internal damping ratio, $\xi_2 = \eta_2/C_r$, called the primary support damping ratio and $\xi_2 = \eta_2/C_r$, called the secondary support damping ratio. Furthermore, the various stiffness ratios have been obtained by dividing by $K$, as given in Appendix 2: i.e., $\beta = K_0/K_1$ is the bearing stiffness ratio, $\beta_2 = K_2/K_1$ is called the primary support stiffness ratio and $\beta_3 = K_3/K_1$ is called the secondary support stiffness ratio.

Equations (8)-(10) are of the form

$$\begin{bmatrix} Z_2 \\ \lambda Z_2 \\ Z_1 \end{bmatrix} = 0.$$  

By putting $|A|=0$, one obtains the characteristic equation for the system as given in Appendix 1.

3. CONDITIONS OF STABILITY

The characteristic equation, given in Appendix 1, can be expressed in the form

$$(a_0 + ib_0)R^6 + (a_1 + ib_1)R^5 + (a_2 + ib_2)R^4 + (a_3 + ib_3)R^3 + (a_4 + ib_4)R^2 + (a_5 + ib_5)R + (a_6 + ib_6) = 0.$$  

(11)

The conditions of stability can be found by applying Routh's criterion to the characteristic equation. One obtains the conditions of stability as follows.

For the $n$th condition

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & 0 \\ b_0 & b_1 & b_2 & \cdots & b_{n-1} & b_n & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & 0 \\ 0 & b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_0 & a_1 & a_n \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & b_n \end{vmatrix} > 0,$$

where the maximum value of $n$ is equal to the highest exponent of the polynomial representing the characteristic equation. As the present characteristic equation is of sixth degree one has six conditions and the maximum order of the determinant would be $12 \times 12$.

For the first condition of stability $(a_1b_0 - a_0b_1) > 0$. Upon putting into it the values of $a_0$, $a_1$, $b_0$ and $b_1$ from the equation in Appendix 1, the left side of this inequality becomes $-(4a_2^2\xi_2^2)(-2a_2^2\xi_2 - 2a_2\beta_2\xi - 8\xi_1\xi_2)$, which is always greater than zero. So the first condition is always satisfied. In order to find the other conditions for stability, a computer program has been written which calculates the value of the determinant corresponding to a condition and checks its sign, for values of $\delta$ increasing in steps. As the sign of the determinant changes and signifies an instability thereby, the program writes down the value of $\delta$ below which the stability is achieved according to the particular condition concerned. In the same way all the six conditions are checked and the stability threshold
for each is obtained. The lowest value obtained from all these conditions is plotted as the stability threshold.

4. RESULTS AND DISCUSSION

A detailed parametric study has been carried out for the system on viscoelastic supports, and results have been compared with those obtained for the cases shown in Figures 1(c) and (d); viz., a viscously damped flexible support corresponding to the Voigt model and a purely elastic support, respectively. For comparison purposes, the parameters $K_1$ or $\eta_1$, i.e., $\beta_1$ or $\xi_1$ (in non-dimensional forms) are kept the same while other parameters are varied. Values of $\delta$ at the threshold of stability have been determined. The system is stable for any value of $\delta$ below the values indicated by the various curves.

In Figure 2 is shown the influence of varying the non-dimensional support mass parameter $a$ for various values of the primary support stiffness ratio $\beta_1$. Over most ranges of $a$, the viscoelastically damped support gives the highest values of $\delta$ at which instability would occur. There is a considerable improvement of the stability threshold with the introduction of damping as compared to the case for purely elastic supports and, except for low values of $\delta$, the increase in support mass tends to reduce the value of $\delta$ at the stability threshold. This was also seen in the work reported in reference [3] for viscously damped supports.

As seen from Figure 3, the values of $\delta$ for viscoelastically damped supports are the highest for all values of bearing stiffness parameter $\beta$. In Figure 4 is shown the effect of the internal damping ratio parameter $\xi$, which is known to have a destabilizing effect on the system. There is a reduction in the highest value of $\delta$ at the stability threshold as $\xi$ increases, for the cases of both viscoelastic and viscous supports, though there is no effect for the case of purely elastic support.

Figure 5 has been drawn to indicate the maximum attainable value of $\delta$ for stability for varying values of the primary stiffness parameter $\beta_1$ and various values of the damping

![Figure 2. Effect of mass ratio $a$ on stability. ——, Viscoelastic model; ------, Voigt model; simplex, Elastic model. $\beta = 0.8$, $\xi = 0.005$, $\beta_1 = 0.2$, $\xi_1 = 0.03$, $\xi_2 = 0.04$.](image-url)
Figure 3. Effect of bearing stiffness ratio $\beta$ on stability. $\alpha=0.8$, $\xi=0.005$, $\beta_2=0.2$, $\xi_1=0.03$, $\xi_2=0.04$. Key as Figure 2.

Figure 4. Effect of internal damping ratio $\xi$ on stability. $\alpha=0.8$, $\beta=8.0$, $\xi_1=0.03$, $\xi_2=0.04$, $\beta_2=0.2$. Key as Figure 2.
parameter $\xi_1$. It is seen that for high values of support stiffness and damping the maximum value of $\delta$ is lowered. However, for medium values of these parameters, values of $\delta$ for stability are high. For purely elastic support, $\delta$ is less than unity and the difference in the maximum value of $\delta$ (which can be attained without the system becoming unstable) between the viscoelastically and viscously damped cases is high for smaller values of $\xi_1$, the difference being almost negligible for $\xi_1 > 0.7$ and the two curves coinciding for $\xi_1 = 2.0$. A peak in the values of $\delta$ is seen to occur for certain values of $\beta_1$ for both these cases. Thus, the introduction of support damping, with proper values of $\xi_1$ and $\beta_1$ is considerably beneficial from the stability point of view, compared to the case of purely elastic supports without damping.

In Figures 6 and 7 the performance of viscoelastically and viscously damped supports are compared for various values of the secondary support parameters $\beta_2$ and $\xi_2$, for
Figure 7. Effect of changing secondary support stiffness ratio $\beta_1$ for varying $\beta_1$ on stability. $a = 0.8$, $\beta = 8.0$, $\xi = 0.005$, $\xi_1 = 0.01$, $\xi_2 = 0.04$.

varying values of $\beta_1$. It is seen that by suitable choice of $\beta_2$ and $\xi_2$ for the case of viscoelastically damped supports, the values of $\delta$ at the stability threshold can be considerably enhanced compared to those for viscously damped support.

5. CONCLUSIONS

From the work described, it can be concluded that viscoelastic supports increase the stability limit compared to that with either viscously damped flexible supports or elastic supports and with proper selection of the values of the support parameters the stability threshold can be increased significantly for a system on viscoelastic supports.

REFERENCES

APPENDIX 1: CHARACTERISTIC EQUATION

\[ 4a\xi_2 R^6 + \left[-4a\xi_2^2R^5 + (-2a\xi_2 - 2a\beta_2 - 2a\beta_2^2 - 2a\beta_2\xi_2 - 8\xi_1\xi_2) R^4 \right. \\
-4\beta_1\xi_2^2 - 4a\beta_2\xi_2^2 + i(2a\beta_2\xi_2^2 + 8\xi_1\xi_2\delta) \right] R^3 \\
+ [4(\beta_1\xi_2^2 + 4\beta_2\xi_2^2 + 4\beta_2\xi_2 + 4\beta_1\xi_2 + 4a\beta_2\xi_2^2) + i(2\beta_2\xi_1 + 2\beta_2\xi_2 + 2\beta_2\xi_1 + 2\beta_2\xi_2 + 2\beta_1\xi_2) \\
+ 2(2\beta_1\xi_1 + 2\beta_1\xi_2 + 2\beta_2\xi_1 + 2\beta_2\xi_2 + 2\beta_1\xi_2) + i(2\beta_1\xi_1 + 2\beta_1\xi_2 + 2\beta_2\xi_1 + 2\beta_2\xi_2) \right] R^2 \\
+ [(-4\beta_2\xi_1^2 - 4\beta_2\xi_2^2 - 4\beta_2\xi_2 + 4\beta_1\xi_2^2 i] R \\
+ \left[(-\beta_1\xi_2^2 - 2\beta_2\xi_2^2 - 2\beta_1\xi_2^2 - 2\beta_1\xi_2^2) \right] = 0. \]

APPENDIX 2: NOTATIONS

- \(C_i\): internal damping coefficient
- \(K\): equivalent stiffness of supports
- \(K_b\): total bearing stiffness
- \(K_i\): stiffness of spring elements, \(i = 1, 2\)
- \(K_s\): stiffness of the shaft at the rotor disk station
- \(M_1\): total support mass
- \(M_2\): mass of the rotor disk
- \(z_1\): complex displacement of the support mass
- \(z_2\): complex displacement of the rotor disk
- \(z_j\): complex displacement of the journal
- \(z_s\): complex shaft relative motion
- \(\alpha = \frac{M_1}{M_2}\): mass ratio
- \(\beta = \frac{K_b}{K_s}\): bearing stiffness ratio
- \(\beta_1 = \frac{K_1}{K_2}\): primary support stiffness ratio
- \(\beta_2 = \frac{K_2}{K_3}\): secondary support stiffness ratio
- \(\eta_i\): damping coefficient in the model, \(i = 1, 2\)
- \(\xi\): internal damping ratio
- \(\xi_1\): primary support damping ratio
- \(\xi_2\): secondary support damping ratio
- \(\omega\): angular speed of the rotor shaft
- \(\omega_n\): natural frequency of the rotor on rigid supports
- \(\delta = \omega / \omega_n\): frequency ratio
- \(\lambda\): variable frequency
- \(R = \lambda / \omega_n\): variable frequency ratio
- \(D\): differential operator
- \((\cdot)\): differentiation with respect to time