Evaluation of generation system reliability indices by fast transform techniques

S R Lakshmi, S C Tripathy, K S P Rao and R Balasubramanian
Indian Institute of Technology, New Delhi, India

Two computationally efficient Fourier transform methods namely, the mixed radix Fast Fourier transform algorithm (FFTA) and the Winograd Fourier transform algorithm (WFTA) for evaluating the reliability of a power system are presented in this paper. Both the methods are used to evaluate the reliability of the IEEE reliability test system and they are compared with the direct discrete Fourier transform method and the basic recursive approach in terms of accuracy and computer time. A sensitivity analysis has been performed by variation of a few system variables to compare the two methods.

Keywords: reliability, LOLP, EENS, Fourier transform

I. Introduction
Reliability is one of the most important criteria which must be taken into account during the design and planning phases of a power system. Reliability evaluation of a power system basically consists of the evaluation of certain reliability indices. The most commonly used indices for reliability assessment of a power system are the loss of load expectation (LOLE) and the expected value of energy not served (EENS). These are useful indices which can be used to compare the adequacy of alternative configurations and expansions. Evaluation of these reliability indices requires the convolution of the outage probability density functions of the generating units of the system to develop the capacity outage probability table, which is then convolved with an appropriate system load characteristic to give an expected risk of loss of load and EENS.

In Reference 2, a methodology has been provided to develop the capacity outage probability table for the generation system by adding the units one by one using a recursive approach. The individual hourly or daily peak loads are used in conjunction with the capacity outage probability table to obtain the LOLE, and the basic expected energy curtailed concept is used to evaluate total EENS of the system. The method is accurate but consumes a considerable amount of computer time in the development of the capacity outage probability table of the system.

The convolution process of the probability density functions of the generating units is simplified in the Fourier domain. The application of the Fourier methods to approximate the probability distribution of outages of generating units was suggested by Schenk and Rau. Computationally, this method is efficient but it is less accurate because it is an approximate method. The Fourier transform method is based on the Gram–Charlier expansion of a distribution. Some approximations are introduced in representing the Fourier transforms of the individual probability outage density functions, the load duration curve and in fitting the Gram–Charlier series to the convolved distribution. This method is equally accurate when compared with the basic recursive method only when the generation system is large and the units have large forced outage rates.

In Reference 4, the base 2 Fast Fourier transform algorithm (base 2 FFTA) was used to simplify the convolution process in the calculation of reliability indices. The method is used to transform the functions to be convolved. The transformed functions are then convolved by using a point by point multiplication and the convolved distribution is inverse transformed to give the LOLE and EENS. The method is much faster than the basic recursive method. It models the true discrete nature of the generating units. It can be applied for large and small systems.

In this paper, two computationally efficient Fourier transform methods namely, the mixed radix Fast Fourier transform algorithm (FFTA) and the Winograd Fourier transform algorithm (WFTA) are proposed for the evaluation of the generation system reliability indices LOLE and EENS. FFTA is an efficient algorithm for computing the Fourier transform. It is a generalized form
of the base 2 FFTA and is hence a versatile method. In FFTA, a mixed radix factorization of the number of sample points \( N \) (used for sampling the probability density functions) is done, whereas in base 2 FFTA, \( N \) can be factorized into powers of 2 only. It is because of this factorization that the FFTA is much faster than the base 2 FFTA.

WFTA developed by Winograd is also an efficient algorithm for computation of the Fourier transform. In WFTA, the number of sample points \( N \) is restricted to a few values. The computation of the Fourier transform by the WFTA requires less number of operations than FFTA, base 2 FFTA and direct discrete Fourier transform. A comparison between the FFTA and WFTA with respect to the direct discrete Fourier transform is given in Section III. Application of these methods to a particular case of generation capacity reliability evaluation of the IEEE reliability test system is discussed in Section IV.

A sensitivity analysis has been performed by varying the system peak load and derated states of the units to compare the two methods.

II. Description of the proposed methods

Discrete probability distributions can be Fourier transformed by using the discrete Fourier transform (DFT). The discrete Fourier transform is represented as

\[
X(n) = \sum_{k=0}^{N-1} x(k)W^{nk} \tag{1}
\]

where \( n = 0, \ldots, N - 1 \), \( W = \exp(-j\pi/N) \), \( \{x(k)\} \) and \( \{X(n)\} \) are complex values. \( \{x(k)\} \) represents the discrete values of the function to be transformed. \( \{X(n)\} \) is the Fourier transform of the function. \( N \) is the number of sample points used to sample the function. For example, if \( N = 4 \), we have \( W = \exp(-j2\pi/4) \) and equation (1) can then be represented in the matrix form as

\[
\begin{bmatrix}
X(0) \\
X(1) \\
X(2) \\
X(3)
\end{bmatrix} =
\begin{bmatrix}
W^0 & W^0 & W^0 & W^0 \\
W^0 & W^1 & W^2 & W^3 \\
W^0 & W^2 & W^0 & W^2 \\
W^0 & W^3 & W^2 & W^1
\end{bmatrix}
\begin{bmatrix}
x(0) \\
x(1) \\
x(2) \\
x(3)
\end{bmatrix} \tag{2}
\]

Solution of the system of equations given in equation (2) gives the discrete Fourier transform \( X(n) \). The evaluation of \( X(n) \) involves \( N^2 \) complex multiplications and \( N(N-1) \) complex additions.

The number of additions and multiplications involved in evaluating DFT by this direct method is very large and consumes a lot of time. The DFT can be evaluated much faster by the following two algorithms: the mixed radix Fast Fourier transform algorithm (FFTA) and the Winograd Fourier transform algorithm (WFTA).

II.1 Mixed radix Fast Fourier transform algorithm

Fast Fourier transform is a particular method of performing a series of computations, that can compute the discrete Fourier transform rapidly. The mixed radix Fast Fourier transform algorithm was developed by Singleton using the factorization of Sande. The algorithm used is described in Appendix I. The basic idea of the algorithm is that of factorizing the number of sample points \( N \) as

\[
N = \prod_{i=1}^{m} N_i \tag{3}
\]

where \( N_i \in \{2, 3, 4, 5 \text{ and odd prime factors up to } 23\} \). The \( N_i \)s may or may not be relatively prime. \( N \) should be chosen such that it can preferably be factorized into factors less than or equal to 5. For choice of factors greater than 5, the number of computations involved is large. FFTA provides for a wide choice of \( N \). The evaluation of DFT of \( N \) points involves evaluation of DFT of each of the factors of \( N \) and combining them according to the algorithm of Sande–Tukey. In evaluation of the DFT of each factor, the formulation of the equations is done such that they involve a minimum number of operations. The number of operations involved for each factor during transform calculations is given in Table 1.

The total number of complex operations involved in computing the DFT is given by the following generalized relation:

\[
M(m; N) = \sum_{i=1}^{m} M_{N_i} - \left[ \sum_{i=2}^{m} \prod_{j=1}^{i-1} N_j \right] \left[ \prod_{j=1}^{i-1} N_j - 1 \right] \tag{4}
\]

\[
C(m; N) = \sum_{i=1}^{m} C_{N_i} \tag{5}
\]

where \( M(m; M) \) and \( C(m; M) \) denote the complex multiplications and complex additions respectively. The number of real operations involved for evaluating DFT for complex data are

\[
M_r = 4M(m; N) \tag{6}
\]

\[
C_r = 2C(m; N) + 2M(m; N) \tag{7}
\]

where \( M_r \) and \( C_r \) are the total number of real multiplications and real additions respectively.

As the total number of operations in evaluating FFTA is less than the direct method, the accuracy would be better and the round off error is reduced.

II.2 Winograd Fourier transform algorithm

The Winograd Fourier transform algorithm, developed by Winograd is an efficient algorithm for rapid computation of the discrete Fourier transform. The basic idea of this algorithm is to factorize \( N \) into a set of Winograd factors belonging to \( \{2, 3, 4, 5, 7, 8, 9, 16\} \). The factors have to be relatively prime. WFTA is restricted to a few values of \( N \). Winograd has developed small algorithms for each factor of the Winograd set for evaluating DFT (equation (1)) and has formulated the
evaluation of generation system reliability indices: S. R. Lakshmi et al.

Table 2. Number of operations in WFTA

<table>
<thead>
<tr>
<th>Factor</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_N</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>8</td>
<td>13</td>
<td>18</td>
</tr>
<tr>
<td>I</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>17</td>
<td>14</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>O</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>9</td>
<td>19</td>
<td>12</td>
<td>24</td>
<td>34</td>
</tr>
</tbody>
</table>

Table 3. Number of operations for FFTA, WFTA and direct method for complex data

<table>
<thead>
<tr>
<th>N</th>
<th>48</th>
<th>120</th>
<th>240</th>
<th>420</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct method</td>
<td>M_N</td>
<td>2304</td>
<td>14400</td>
<td>57600</td>
</tr>
<tr>
<td></td>
<td>C_N</td>
<td>2256</td>
<td>14280</td>
<td>57360</td>
</tr>
<tr>
<td>FFTA</td>
<td>M_N</td>
<td>292</td>
<td>1392</td>
<td>2980</td>
</tr>
<tr>
<td></td>
<td>C_N</td>
<td>754</td>
<td>3118</td>
<td>6834</td>
</tr>
<tr>
<td>WFTA</td>
<td>M_N</td>
<td>108</td>
<td>288</td>
<td>648</td>
</tr>
<tr>
<td></td>
<td>C_N</td>
<td>636</td>
<td>2076</td>
<td>5016</td>
</tr>
</tbody>
</table>

Order of factors in WFTA: 16, 3, 5, 7, 3, 4

The total number of complex operations involved in computing the DFT is given by the following generalized relation:

\[ M(N) = \prod_{i=1}^{m} M_{N_i} \]  

Number of additions:

\[ C(N) = \sum_{i=1}^{m} \left( \prod_{j=1}^{n-1} N_j \right) (N_N + a_N) \left( \prod_{j=1}^{m-n-1} M_N \right) \]

where \( i = m + 1 - l \) and

\[ \prod_{i=a}^{b} (-1) = 1 \] for \( a > b \)

For complex data:

number of real multiplications: \( M_N = 2M(N) \)

number of real additions: \( C_N = 2C(N) \)

III. Comparison of FFTA and WFTA with the direct method for DFT evaluation

To assess the actual gain in computations in FFTA and WFTA one needs to know the ratio of the time taken for one real multiplication to the time taken for one real addition. This factor is different for different systems and is particularly large for many systems where the savings in multiplications is more effective. The complexity of the algorithm also tends to slow down its software implementation.

Both FFTA and WFTA are basically algorithms to compute the DFT more rapidly. They compute the DFT with equal accuracy in comparison with the direct method. Whenever the choice of \( N \) is such that it is factorizable for both FFTA and WFTA, WFTA should be used, as it involves less operations, hence less computer time is required. But when \( N \) is factorizable for FFTA only, then the FFTA can be more efficiently used for DFT computations than the direct method. Table 3 gives the number of operations for FFTA, WFTA and the direct method.

It can be clearly seen from Table 3 that FFTA and WFTA are much faster than the direct discrete Fourier transform calculations.

IV. Calculation of LOLP and EENS using FFTA and WFTA

The calculation of the reliability indices LOLP and EENS of a power system using the FFTA and WFTA is described in this section in a generalized algorithm form.

Step 1

The capacity values at which the generating unit outage functions have impulses are read after sampling by \( N \) points. A complex function is defined where the real part represents the impulses of the outage density functions and the imaginary part is put to zero.

Step 2

This complex function is transformed using the FFTA or WFTA.

Step 3

If there are \( J \) identical units the transformed outage density functions of all such units taken together is obtained by raising the transform of the single unit to \( J \).

Step 4

The transformed outage density functions of all the different units are computed and they are multiplied point by point. This represents convolution in the capacity domain. The product evaluated is the transform of the system outage density function.

Step 5

The load duration curve LDC is also sampled for \( N \) points and a complex function is defined as in step 1 above.

Step 6

The LDC complex function is transformed by FFTA or WFTA. The LDC transform is multiplied point by point with the system outage density transform.

Step 7

The product computed in step 6 is inverse transformed by the FFTA or WFTA.

Step 8

A real function is obtained which is called the equivalent load duration curve. The loss of load probability (LOLP) is read using the equivalent load duration curve. The LOLP is evaluated as the product of LOLP and the time period under consideration. The expected value of energy not served (EENS) is obtained by computing the area under the curve to the right of the abscissa value of system installed capacity by the trapezoidal rule.
The number of points needed to discretize the distribution is such that the sampling interval chosen should be such that it divides almost all the unit capacity. For example, when \( T = 10 \text{ MW} \), \( N > 626 \) points. The generating units are sampled quite accurately, but the LDC sampling is not accurate. Though the computational effort is reduced, accuracy is reduced due to inaccurate sampling of LDC. For \( T = 1 \text{ MW} \), \( N > 6255 \) points. Both the generating units and LDC are sampled exactly. The required accuracy is obtained but computational effort involved is large. Hence, \( T \) is chosen as 5 MW making a compromise between time and accuracy.

Choosing a value of \( N = 1260 \), the nearest possible value to 1251 points, which is factorizable for both FFTA and WFTA, the results for LOLE, EENS and the computational times are given in Table 5 for FFTA, WFTA and the direct method.

The values of LOLP and EENS evaluated by recursive method are 9.394 h/yr and 1176.0 MW h/yr. The computational time involved is 440 ms.

Because the IEEE-RTS contains large number of units of different capacities, many capacity outage states are possible. These states can be reduced by using the Fast transform method. The effect of variation of step size on the LOLE is studied to compare the Fast transform methods and recursive method (with rounding) with respect to accuracy. The results are given in Table 6. The choice of the Fast transform method depends on the value of \( N \). If the nearest possible value of \( N \) chosen is such that it is factorizable for both FFTA and WFTA, then WFTA is used. If \( N \) is such that it is factorizable for FFTA only, then FFTA is used.

Table 6 shows that the accuracy of the Fast transform method is comparable with the recursive method, while the computing time is much less as the time taken for formulation of the capacity outage probability table (COPT) by the recursive method. Depending upon the accuracy required, the choice of capacity rounding increment or step size is made.

The effect of variation of step size on the LOLE is given in Table 7 and Table 8 respectively.

### Numerical results

The algorithms FFTA, WFTA, direct DFT method and the recursive approach are applied to the IEEE reliability test system. This is a 32-machine system with a total capacity of 3405 MW and system peak load of 2850 MW. Table 4 gives information regarding the generating units of IEEE reliability test system.

To compute the discrete Fourier transform of the outage probability density functions of the generating units and the load duration curve of the system by the algorithms, it is necessary to discretize these probability distributions. To discretize these distributions, it is necessary to make a choice of the sampling interval \( T \) and to compute the total length of the capacity domain \( T_n \). The sampling interval \( T \) is evaluated as the sum of the total installed capacity and the peak load. A proper sampling interval has to be chosen depending on the size of the units. \( T \) should be such that it divides almost all the unit capacity values. The unit outage impulses not exactly divisible by \( T \) are represented according to Reference 4. Thus, the number of points needed to discretize the distribution is chosen as

\[
N \geq \frac{T_n}{T}
\]  

The total length of the capacity domain = 6255 MW, capacity sampling interval chosen = 5 MW, no. of points should be greater than \( = \frac{6255}{5} = 1251 \).

The sampling interval chosen should be such that it accurately samples both the generating unit probability density functions and the load duration curve (LDC). The sampling interval 5 MW chosen almost divides all the unit capacities and it also samples the LDC quite accurately. Depending upon \( T \), rounding errors may also be introduced during interpolation while sampling of generating unit probability density functions. Since LDC is a continuous distribution, a larger sampling interval chosen would inaccurately sample the LDC. For example, when \( T = 10 \text{ MW} \), \( N > 626 \) points. The generating units are sampled quite accurately, but the LDC sampling is not accurate. Though the computational effort is reduced, accuracy is reduced due to inaccurate sampling of LDC. For \( T = 1 \text{ MW} \), \( N > 6255 \) points. Both the generating units and LDC are sampled exactly. The required accuracy is obtained but computational effort involved is large. Hence, \( T \) is chosen as 5 MW making a compromise between time and accuracy.

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Table 6 shows that the accuracy of the Fast transform method is comparable with the recursive method, while the computing time is much less as the time taken for formulation of the capacity outage probability table is very large in the case of the recursive method.

A sensitivity analysis has also been carried out to study the impact of the variation of system peak load and derated states on LOLE and EENS for comparison of the two methods. The effect of variation of system peak load and derated states of the generating units on the LOLE and EENS is given in Table 7 and Table 8 respectively.
Table 7. Effect of variation of system peak load on LOLE and EENS for FFTA and WFTA

<table>
<thead>
<tr>
<th>System peak load (MW)</th>
<th>$N$ greater than</th>
<th>$N \leq 5$</th>
<th>FFTA</th>
<th>WFTA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LOLE</td>
<td>EENS</td>
<td>LOLE</td>
</tr>
<tr>
<td>2350</td>
<td>1151</td>
<td>1152</td>
<td>0.19</td>
<td>16.7</td>
</tr>
<tr>
<td>2550</td>
<td>1191</td>
<td>1200</td>
<td>1.09</td>
<td>112.2</td>
</tr>
<tr>
<td>2750</td>
<td>1232</td>
<td>1250</td>
<td>4.87</td>
<td>563.9</td>
</tr>
<tr>
<td>2850</td>
<td>1251</td>
<td>1280</td>
<td>9.41</td>
<td>1175.1</td>
</tr>
<tr>
<td>2950</td>
<td>1271</td>
<td>1296</td>
<td>17.56</td>
<td>2325.0</td>
</tr>
<tr>
<td>3150</td>
<td>1311</td>
<td>1350</td>
<td>53.19</td>
<td>7980.3</td>
</tr>
<tr>
<td>3350</td>
<td>1351</td>
<td>1440</td>
<td>136.86</td>
<td>13977.4</td>
</tr>
</tbody>
</table>

Table 8. Derated states for the units

<table>
<thead>
<tr>
<th>Unit (MW)</th>
<th>States</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>0</td>
<td>0.846154</td>
</tr>
<tr>
<td>200</td>
<td>0.076923</td>
<td></td>
</tr>
<tr>
<td>350</td>
<td>0</td>
<td>0.898438</td>
</tr>
<tr>
<td>175</td>
<td>0.046875</td>
<td></td>
</tr>
<tr>
<td>350</td>
<td>0.054687</td>
<td></td>
</tr>
</tbody>
</table>

Table 9. Effect of derated states on LOLE and EENS for FFTA and WFTA ($N = 1260$)

<table>
<thead>
<tr>
<th>Unit</th>
<th>FFTA</th>
<th>WFTA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $\times$ 400</td>
<td>7.78</td>
<td>940.82</td>
</tr>
<tr>
<td>2 $\times$ 400</td>
<td>6.33</td>
<td>743.91</td>
</tr>
<tr>
<td>2 $\times$ 400 + 1 $\times$ 350</td>
<td>5.67</td>
<td>650.34</td>
</tr>
</tbody>
</table>

V. Conclusions

In this paper, two computationally efficient methods, the mixed radix Fast Fourier transform algorithm and the Winograd Fourier transform algorithm have been proposed for evaluating the reliability of a power system. It is observed that the WFTA is faster than the FFTA for those values of the sample points $N$ which are factorizable for both FFTA and WFTA. For those values of $N$ which can be factorized for FFTA only, FFTA can be used efficiently for reliability evaluation. The two methods are quite accurate and are faster than the basic recursive approach and the direct Fourier transform. Thus, they can be efficiently used for reliability evaluation of power systems.

VI. References


Appendix I

Sande–Tukey algorithm (S–T)

Let the number of points to be discretely transformed satisfy \( N = N_1 N_2 \cdots N_m \) where \( N_1, N_2, \ldots, N_m \) are integer values.

The indices \( n \) and \( k \) in equation (1) can be expressed in the variable radix expression of the form:

\[
\begin{align*}
  n &= n_{m-1} (N_2 N_3 \cdots N_m) + \cdots + n_1 N_2 + n_0 \\
  k &= k_{m-1} (N_2 N_3 \cdots N_m) + \cdots + k_1 N_2 + k_0
\end{align*}
\]  

(AI.1)

where

\[
\begin{align*}
  n_{i-1} &= 0, 1, \ldots, N_i - 1, \quad 1 \leq i \leq m \\
  k_i &= 0, 1, \ldots, N_m - 1, \quad 0 \leq i \leq m - 1
\end{align*}
\]

Equation (1) can then be rewritten as

\[
X(n_{m-1}, \ldots, n_0) = \sum_{k_{m-1}} \sum_{k_{m-2}} x_0(k_{m-1}, \ldots, k_0) W^{nk}
\]

(AI.2)

If the product \( nk \) is expanded in terms of \( n \), we get the S–T algorithm. We have

\[
W^{nk} = W^{nk_0} W^p
\]

where

\[
p = \{n_1 N_1^2 + \cdots + n_{m-1} (N_2 \cdots N_m)\} k
\]

The general form of the recursive arrays in equation (AI.2) takes the form

\[
\tilde{X}(n_0, \ldots, n_{m-1}, k_0) = \left[ \sum_{k_{m-1}} \sum_{k_{m-2}} x_0(k_{m-1}, \ldots, k_0) W^{p_1} \right] W^{p_2}
\]

(AI.3)

where

\[
\begin{align*}
  p_1 &= n_{i-1} k_{m-1} - \left( \frac{N_i}{N_j} \right) \\
  p_2 &= n_{i-1} k_{m-1} + (N_{i+2} \cdots N_m) + \cdots + k_0
\end{align*}
\]

The term \( W^{p_2} \) outside the brackets is called the rotation factor. The splitting up of the power \( p \) into \( p_1 \) and \( p_2 \) reduces the number of multiplications in computing the bracketed term and allows one to take advantage of the symmetry of the sine and cosine functions. For example when \( N = 8 \), \( W \) takes the values,

\[
\pm 1, \pm j, \pm e^{\pi/4}, \pm e^{-j\pi/4}
\]

The first two factors require no multiplications and the product of a complex number of the last two factors requires only two real multiplications. A total of four real multiplications are required to evaluate each eight-point transform.

Appendix II

Winograd Fourier transform algorithm

Let \( N = N_1 N_2 \) where \( N_1 \) and \( N_2 \) are relatively prime. In equation (1) for cyclic convolution the matrix \( W^{nk} \) of complex exponentials needs to be permuted such that the resulting matrix can be partitioned into blocks of \( N_2 \times N_2 \) cyclic matrices and such that the blocks form an \( N_1 \times N_1 \) cyclic matrix. The rearrangement can be done by using the Chinese Remainder (C–R) theorem.

According to this theorem, we can represent every integer, \( 0 \leq i \leq N \), by the pair \((i_1, i_2)\).

For example, because \( 6 = 2 \times 3 \), by the C–R theorem we have the following correspondence:

\[
\begin{align*}
  0 &\rightarrow (0, 0); \quad 1 \rightarrow (1, 1); \quad 2 \rightarrow (0, 2); \quad 3 \rightarrow (1, 0) \\
  4 &\rightarrow (0, 1); \quad 5 \rightarrow (1, 2)
\end{align*}
\]

For cyclic convolution, we must arrange it in the order \( 0, 4, 2, 3, 1, 5 \). Equation (1) can be written as

\[
\begin{bmatrix}
  X_0 \\
  X_1 \\
  X_2 \\
  X_3 \\
  X_4 \\
  X_5
\end{bmatrix} =
\begin{bmatrix}
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix}
\]

(AII.1)

We can write equation (AII.2) as

\[
\begin{bmatrix}
  X_0 \\
  X_4 \\
  X_2 \\
  X_3 \\
  X_1 \\
  X_5
\end{bmatrix} =
\begin{bmatrix}
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  x_4 \\
  x_2 \\
  x_3 \\
  x_1 \\
  x_5
\end{bmatrix}
\]

(AII.2)

Equation (AII.2) is the same as equation (AII.1) but it now exhibits block structure. Defining

\[
Z_0 = X_0, \quad Z_1 = X_1, \quad Y_0 = X_4, \quad Y_1 = X_2
\]

\[
W_0 = \begin{bmatrix}
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\
  W^0 & W^0 & W^0 & W^0 & W^0 & W^0
\end{bmatrix}
\]

We can write equation (AII.2) as

\[
\begin{bmatrix}
  Z_0 \\
  Z_1
\end{bmatrix} =
\begin{bmatrix}
  W_0 & W_0 \end{bmatrix}
\begin{bmatrix}
  Y_0 \\
  Y_1
\end{bmatrix}
\]

(AII.3)

we get

\[
M_1 = W_0 (Y_0 + Y_1)
\]

\[
M_2 = W_0 (Y_0 - Y_1)
\]

\[
Z_0 = M_1, \quad Z_1 = M_2
\]  

(AII.4)
Then the three point convolution is applied. The final array obtained has to be rearranged to obtain the final solution.

$N = 6$ could have factored as $3 \times 2$ resulting in three point convolution of $2 \times 3$ blocks. Winograd has developed algorithms for computing discrete Fourier transform of $2, 3, 4, 5, 7, 8, 9$ and $16$ points. These algorithms involve least number of multiplications. When $N$ is large, $N$ can be factorized into the Winograd factors and the algorithms of Winograd can be used to compute the discrete Fourier transform.