On the design of efficient second and higher degree FIR digital differentiators at the frequency $\pi/(\text{any integer})$

Hitendra Shah, S.C. Dutta Roy

Department of Electrical Engineering, Indian Institute of Technology, Delhi, Hauz Khas, New Delhi 110 016, India

Balbir Kumar

Department of Electronics and Communication Engineering, Delhi Institute of Technology, Kashmere Gate, Delhi 110 006, India

Received 14 January 1991
Revised 22 August 1991

Abstract. In a number of signal processing applications, digital differentiators (DD) of degree greater than unity performing over a narrow band of frequencies are required. The minimax relative error DDs are especially suitable for broad band applications, but they become inefficient when adopted for narrow band situations. This paper proposes second and higher degree DDs which are maximally accurate at the spot frequency: $\pi/(\text{any integer})$. Mathematical relations have been established between the weighting coefficients of the first degree FIR digital differentiators which are maximally linear at the frequency $\pi/(\text{any integer})$ and those of the proposed (second and higher degree) differentiators. It has been shown that very high accuracies in the frequency response of the approximation are achievable with attractively low order of the structure for the suggested differentiators. As an example, with just 16 multiplications per input sample of the signal, it is possible to obtain a third degree differentiator over a frequency bandwidth of 0.20$\pi$ centred around $\omega = \pi/3$, with an accuracy no worse than 99.999%. The phase error is zero over the entire frequency band $0 \leq \omega \leq \pi$ of operation.

Zusammenfassung. In verschiedenen Signalverarbeitungsaufgaben werden digitale Differenziatoren (DD) gebraucht, welche einen Grad größer als eins haben, und in einem schmalen Frequenzband arbeiten müssen. DDs mit einem minimax relativen Fehler sind geeignet für Breitbandanwendungen, werden jedoch ineffizient in Schmalbandsituationen. In dieser Arbeit werden DDs zweiten und höheren Grades eingesetzt, welche maximale Genauigkeit bei einer bestimmten Frequenz der Form $\pi/(\text{ganze Zahl})$ haben. Mathematische Beziehungen zwischen den Gewichten der DDs ersten Grades welche maximal linear bei dieser Frequenz sind und denjenigen der hier eingeführten DDs werden hergeleitet. Es wird gezeigt, dass eine hohe Genauigkeit des Frequenzgangs erreicht werden kann mit erstaunlich niedrigem Grad. Zum Beispiel ist es möglich, mit nur 16 Multiplikationen pro Eingangswert über eine Bandbreite von 0.20$\pi$ um $\omega = \pi/3$ einen Differenziator dritten Grades mit einer Genauigkeit von mindestens 99.999% zu erhalten. Der Phasenfehler ist null über das ganze Frequenzband $0 \leq \omega \leq \pi$.

Résumé. Un grand nombre d'applications en traitement du signal requièrent des différentiateurs numériques (DD) de degré supérieur à l'unité et opérant dans une bande de fréquence étroite. Les DD à erreur relative minimax sont particulièrement bien adaptés à des applications de large bande mais deviennent inefficaces lorsqu'on les adopte pour des applications de bande étroite. Cet article propose des DD de degré deux et plus qui sont précis maximale à la fréquence cible $\pi/(\text{un entier quelconque})$. Nous établissons des relations mathématiques entre les coefficients de pondération des différentiateurs numériques FIR du premier degré qui sont maximale linéaires à la fréquence $\pi/(\text{un entier quelconque})$ et ceux des différentiateurs proposés (degré deux et plus). Nous montrons que des précisions très élevées de la réponse en fréquence de l'approximation sont atteignables avec un ordre attractivement peu élevé de la structure pour les différentiateurs suggérés. Citons comme exemple qu'avec seulement seize multiplications par échantillon d'entrée du signal, il est possible d'obtenir un différentiateur du troisième degré dans une bande de fréquence de 0.20$\pi$ centrée autour de $\omega = \pi/3$, avec une précision au moins aussi bonne que 99.999%. L'erreur de phase est nulle dans toute la bande de fréquence $0 \leq \omega \leq \pi$ d'opération.

Keywords. Differentiators, digital signal processing.
1. Introduction

In a number of data processing systems, higher degree derivatives of the signal are often required for further use. In the analysis of seismic and geological data [1, 6], for example, we require second and higher degree derivatives besides the first one. In the case of radar signals, where the target course is not, in general, a straight line, the use of first and higher degree derivatives of the echo returns (to obtain the target velocity and acceleration, for example), help us to reduce errors in its tracking [9]. The use of higher derivatives of the data in biomechanics is well known. Also, in systems where automatic detection and tracking (ADT) is adopted for target detection and for track initiation, association, smoothing (filtering) and termination [10], we require precise values of differentiation of the available data at or around spot frequencies: \( \omega = \frac{\pi}{p} \), where \( p \) is a positive integer.

The frequency response of an ideal differentiator of \( k \)th degree is given by

\[
\tilde{H}^{(k)}(\omega) = (j\omega)^k = \begin{cases} \tilde{H}^{(k)}(\omega), & k \text{ even}, \quad -\pi \leq \omega \leq \pi, \\ j\tilde{H}^{(k)}(\omega), & k \text{ odd}, \quad -\pi \leq \omega \leq \pi, \end{cases}
\]

where \( \tilde{H}^{(k)}(\omega) \) is purely real and is defined by

\[
\tilde{H}^{(k)}(\omega) = \begin{cases} (-1)^{k/2}(\omega)^k, & k \text{ even}, \\ (-1)^{(k-1)/2}(\omega)^k, & k \text{ odd}. \end{cases}
\]

Rabiner and Schafer [7] have proposed minimax relative error (MRE) digital differentiators (DDs) which are highly suitable for wideband operation. Our earlier work [2-4] dealt with first degree DDs suitable for low, midband and high frequency ranges. Also, FIR digital differentiators of second and higher degrees for low frequencies having maximal accuracy of magnitude response (in the Butterworth sense) at \( \omega = 0 \) were suggested in [8].

In this paper, we extend the concept of [8] for second and higher degree DDs performing around the spot frequency \( \omega = \frac{\pi}{p} \), where \( p \) is a positive integer. We shall denote such an approximation by \( H_p^{(k)}(\omega) \), where the superscript \( k \) indicates the degree of the differentiator.

2. The design

We recall two of our earlier results. First, a digital differentiator of first degree, which is maximally linear at \( \omega = \frac{\pi}{2} \), is approximated by [4]

\[
H_{1}^{(1)}(\omega) = \frac{\pi}{2} \sum_{r=1,0}^{n-1} d_{1}^{(1)} \sin i\omega - \frac{1}{2} \sum_{i=2,0}^{n} d_{1}^{(1)} \sin i\omega, \quad n \text{ even},
\]

where \( n = (N-1)/2 \) is the total number of weights and \( N \) is the order of the approximation. The weights \( d_{1}^{(1)} \) are given by [4]

\[
d_{1}^{(1)} = \left[ \left( \frac{i-2}{(i-1)/2} \right) \left[ \frac{2^{i-3}-s_{0}}{2^{i-3}} \right] \right]
\]

\[
+ \sum_{k=+(i+1)/2}^{(n-3)/2} (-1)^{(2k+i+1)/2} \left( \frac{2k+i-1/2}{2k-i+1/2} \right) d_{1}^{(1)},
\]

\[
i = n-1, n-3, n-5, \ldots, 5, 3, 1
\]

(descending order),

\( i \) odd,

\[
d_{1}^{(1)} = \left[ \left( \frac{i-1}{i/2} \right) \right]
\]

\[
+ \sum_{r=1}^{(n-1)/2} (-1)^{r+1} \left( \frac{i+r-1}{r} \right) d_{2}^{(1)} d_{2}^{(1)},
\]

\[
i = n, n-2, n-4, \ldots, 6, 4, 2
\]

(descending order),

\( i \) even.

\[\text{(3a)}\]

\[\text{(3b)}\]
An even value of \( n \) has been assumed in (2) for the reasons given in [4].

Secondly, a digital differentiator of first degree which is maximally linear at \( \omega = \pi \), is approximated by [3]

\[
H_{p}^{(1)}(\omega) = \pi \sum_{i=1}^{n-1} a_i \sin i \omega + \sum_{i=1}^{n-2} b_i \sin i \omega, \\
n = (N - 1)/2, \quad n \text{ even,} \quad (4)
\]

where \( n \) gives the total number of weights \(^2\) (a\(_i\)’s and b\(_i\)’s). The formulas giving \( a_i \) and \( b_i \) are available in [3]. By simple algebraic manipulations, (4) can be written as

\[
H_{p}^{(1)}(\omega) = \pi \sum_{i=1, i \text{ odd}}^{n-1} a_{i+1/2} \sin \frac{i \omega}{2} + \sum_{i=2, i \text{ even}}^{n} b_{i/2} \sin \frac{i \omega}{2}, \quad n \text{ even.} \quad (5)
\]

It can be shown, after considerable algebraic operations and by using some combinatorial identities, that

\[
a_{i+1/2} = d_{i}^{(1)}, \quad i \text{ odd,} \quad (6a)
\]
\[
b_{i/2} = -d_{i}^{(1)}, \quad i \text{ even.} \quad (6b)
\]

Using the results (6) in (5), we may combine the resulting equation with (2) to obtain the following composite form:

\[
H_{p}^{(1)}(\omega) = \frac{1}{p} \left[ \pi \sum_{i=1, i \text{ odd}}^{n-1} d_{i}^{(1)} \sin \frac{i \omega}{2} - \sum_{i=2, i \text{ even}}^{n} d_{i}^{(1)} \sin \frac{i \omega}{2} \right], \\
p = 1, 2, \ldots, n \text{ even.} \quad (7)
\]

We have shown [5] that \( H_{p}^{(1)} \) given by (7) represents an approximation of \( \tilde{H}_{p}^{(1)}(\omega) \) such that it is maximally linear at the spot frequency \( \omega = \pi/p \), where \( p \) is any positive integer (and not necessarily 1 or 2 only).

2.1. DD of second degree \((H_{p}^{(2)}(\omega))\)

For an ideal DD of second degree, we get, from (1b),

\[
\tilde{H}_{p}^{(2)}(\omega) = (-1)^{2/2} \omega^{2} = -\omega^{2}, \quad -\pi \leq \omega \leq \pi. \quad (8)
\]

Since \( -\omega^{2} = -2 \int \omega \, d\omega + \text{(constant of integration)} \), we may replace \( \omega \) and \( -\omega^{2} \) by their respective approximations [8] and obtain \( H_{p}^{(2)}(\omega) = -\int H_{p}^{(1)}(\omega) \, d\omega + d_{0}^{(2)} \), where \( d_{0}^{(2)} \) is the constant of integration. Thus, integrating (7), we get

\[
H_{p}^{(2)}(\omega) = \frac{2}{p} \left[ \pi \sum_{i=1, i \text{ odd}}^{n-1} \frac{2 d_{i}^{(1)} \cos \frac{i \omega}{2}}{ip} - \sum_{i=2, i \text{ even}}^{n} \frac{2 d_{i}^{(1)} \cos \frac{i \omega}{2}}{ip} \right] + d_{0}^{(2)}, \quad (9a)
\]

\[
= \frac{1}{p} \left[ \pi \sum_{i=1, i \text{ odd}}^{n-1} d_{i}^{(2)} \cos \frac{i \omega}{2} - \sum_{i=2, i \text{ even}}^{n} d_{i}^{(2)} \cos \frac{i \omega}{2} \right] + d_{0}^{(2)}, \quad (9b)
\]

where

\[
d_{i}^{(2)} = 2 \left( \frac{2 d_{i}^{(1)}}{ip} \right), \quad i = 1, 2, \ldots, n. \quad (10)
\]

We also let the approximation \( H_{p}^{(2)}(\omega) \) have ideal frequency response at \( \omega = \pi/p \), i.e.,

\[
H_{p}^{(2)}(\omega) \big|_{\omega = \pi/p} = -(\pi/p)^{2}. \quad (11)
\]

By using (9b) and (11), we obtain

\[
d_{0}^{(2)} = -\left( \frac{\pi}{p} \right)^{2} + \frac{1}{p} \sum_{i=2, i \text{ even}}^{n} (-1)^{i} d_{i}^{(2)}. \quad (12)
\]

Equations (10) and (12) give all the weighting coefficients required for the approximation.

\[^2\] Here, we have designated the weights as \( a_i \) and \( b_i \) instead of \( c_i \) and \( d_i \) used in [3] to avoid confusion with \( d_i \) also used in (2) of this paper.
Fig. 1. Frequency response, $H_2^{(2)}(\omega)/\pi^2$, of second degree differentiators maximally accurate at $\omega = \pi/3$ for $n = 2, 6, 10, 16$ and 32.

Fig. 2. Frequency response, $H_2^{(3)}(\omega)/\pi^2$, of second degree differentiators maximally accurate at $\omega = \pi/4$ for $n = 2, 6, 10, 16$ and 32.

$H_2^{(2)}(\omega)$, since $d_{1}^{(1)}$ can readily be computed from (3).

$H_2^{(3)}(\omega)$, since $d_{0}^{(3)}$ can be computed from (3).

$\tilde{H}^{(3)}(\omega) = (-1)^{3-1}/2\omega^3 = -\omega^3$.

$-\pi \leqslant \omega \leqslant \pi$. (13)

2.2. DD of third degree ($H_2^{(3)}(\omega)$)

For the ideal DD of third degree, we have, from (1b),

By using the methodology of the previous case, we may write $H_2^{(3)}(\omega) = 3 \int H_2^{(2)}(\omega) \, d\omega + d_0^{(3)}$ (a constant). By using (9b), (10) and (12), it is then easy
Fig. 3. Frequency response, $H_1^{(3)}(\omega)/\pi^3$, of third degree differentiators maximally accurate at $\omega = \pi/3$ for $n = 2, 6, 10, 16$ and 32.

Fig. 4. Frequency response, $H_2^{(3)}(\omega)/\pi^3$, of third degree differentiators maximally accurate at $\omega = \pi/4$ for $n = 2, 6, 10, 16$ and 32.

to see that

$$H_p^{(3)}(\omega) = \frac{1}{p} \left[ \pi \sum_{i=1}^{n-1} d_i^{(3)} \sin \frac{ip\omega}{2} - \sum_{i=2}^{n} d_i^{(3)} \sin \frac{ip\omega}{2} \right] + d_0^{(3)}; \quad i = 1, 2, \ldots, n$$

(14)

where

$$d_i^{(3)} \triangleq 3 \left( \frac{2d_i^{(2)}}{ip} + d_0^{(2)}d_i^{(1)} \right), \quad i = 1, 2, \ldots, n$$

(15)

and

$$d_0^{(3)} = -\left( \frac{\pi}{p} \right)^3 + \frac{1}{p} \sum_{i=1}^{n-1} (-1)^{i+1/2} d_i^{(3)}.$$  

(16)
Proceeding thus, we arrive at the following general expression for $H_p^{(k)}(\omega)$:

$$H_p^{(k)}(\omega) = \begin{cases} 
\frac{1}{p} \left[ \pi \sum_{i=1, i \text{ odd}}^{n-1} d_i^{(k)} \sin \frac{ip\omega}{2} \right] - \sum_{i=2, i \text{ even}}^{n} d_i^{(k)} \sin \frac{ip\omega}{2}, \\
\frac{1}{p} \left[ \pi \sum_{i=1, i \text{ odd}}^{n-1} d_i^{(k)} \cos \frac{ip\omega}{2} \right] - \sum_{i=2, i \text{ even}}^{n} d_i^{(k)} \cos \frac{ip\omega}{2}, \\
k=1, 3, 5, \ldots, \\
k=2, 4, 6, \ldots. 
\end{cases}$$  

(17)  

(18)

The general relations connecting the weighting coefficients for DDs of degree $k$ to the corresponding coefficients of those of degrees $k-1$ and 1 are given in Table 1.

3. Performance

Figures 1 and 2 show the frequency response curves for second degree differentiators $H_p^{(2)}(\omega)$ and $H_p^{(2)}$ for $p=3$ and 4, respectively. Figures 3 and 4 give the corresponding curves for the third degree differentiators $H_p^{(3)}(\omega)$ and $H_p^{(3)}$. We define the relative error (RE) of approximation $H_p^{(k)}$ as

$$RE = \frac{|H_p^{(k)}(\omega)| - |\hat{H}_p^{(k)}(\omega)|}{|\hat{H}_p^{(k)}(\omega)|}. \quad (19)$$

Tables 2 and 3 give the number of multipliers, $n$, per input sample of the signal for REs less than or
Table 2
Number of multiplications, $n$, of the second degree digital differentiators for different relative errors at frequency $\pi/p$ for $p = 3$ and 4

<table>
<thead>
<tr>
<th>Frequency range $\omega/p$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega \leq -100$ dB</td>
<td>$\omega \leq -80$ dB</td>
<td>$\omega \leq -60$ dB</td>
</tr>
<tr>
<td>$1/p \pm 0.025$</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$1/p \pm 0.05$</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$1/p \pm 0.075$</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>$1/p \pm 0.100$</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 3
Number of multiplications, $n$, of the third degree digital differentiators for different relative errors at frequency $\pi/p$ for $p = 3$ and 4

<table>
<thead>
<tr>
<th>Frequency range $\omega/p$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega \leq -100$ dB</td>
<td>$\omega \leq -80$ dB</td>
<td>$\omega \leq -60$ dB</td>
</tr>
<tr>
<td>$1/p \pm 0.025$</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>$1/p \pm 0.05$</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>$1/p \pm 0.075$</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>$1/p \pm 0.100$</td>
<td>16</td>
<td>14</td>
</tr>
</tbody>
</table>

equal to $-100$ dB, $-80$ dB, $-60$ dB and $-40$ dB. It may be seen that for $p = 4$, a second degree differentiator, $H_4^{(2)}(\omega)$, requires only 14 multiplications per input sample of the signal for $\omega$ less than or equal to $-100$ dB (i.e., an accuracy $\geq 99.999\%$) for the frequency range $0.15\pi \leq \omega \leq 0.35\pi$. Had we cascaded two first degree differentiators (i.e., used two $H_4^{(1)}(\omega)$s) to obtain the second degree differentiator $H_4^{(2)}(\omega)$, we would have required 24 multiplications per input sample for comparable performance.

Similarly, a third degree DD as proposed by us, requires 20 multiplications against 36 required in the 'cascaded' approach for $p = 4$ for the aforementioned specifications. It is observed that very low relative errors are indeed achievable from the proposed design of digital differentiators for attractively low orders, operating efficiently at and around the spot frequencies $\omega = \pi/(\text{any integer})$.

The exact mathematical formulas, as derived in this paper, are preferred over algorithmic approaches (see [7], for example) for calculation of the weighting coefficients where the speed of computation is the prime concern. Very high accuracy of the frequency response of the approximation, zero phase error, narrow frequency bands of operation, ability to perform over spot frequencies: $\omega = \pi/(\text{any integer})$ and the economy of the structure are some of the attractive features of the proposed design.

4. Conclusions

We have proposed here efficient, FIR, second and higher degree digital differentiators, which are suitable for operation around spot frequencies: $\omega = \pi/(\text{any integer})$, with very low relative errors in the frequency response. Mathematical formulas for calculation of the weighting coefficients have also been derived.

Acknowledgment

The authors thank the reviewers for their constructive suggestions and useful comments.
References


