Probabilistic analysis of subsynchronous resonance in series-compensated EHV transmission lines

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The stability limit curves in the $R-X$ parametric plane have been determined for EHV lines with series-compensation taking into consideration the nondeterministic nature of the generating system, mechanical system, and transmission system parameters. Studies have shown that the effect of random system parameters is to increase the area of the unstable zone in the parametric plane. First, the border line between the stable and unstable zones has been determined considering a deterministic model of the system. The value of the critical frequency at which the self-excited oscillations including the effect of hunting are set up and the system critical resistance necessary to damp out the oscillations are obtained using a modified D-partition technique. The modification involves the determination of the critical frequency by the Newton method. The nature of the random system parameters is then considered using the results of the deterministic study to calculate the variances of the critical frequency and resistance, and the stability limit curves.

Keywords: subsynchronous resonance, D-partition technique, short-term system dynamics, dynamic stability, probabilistic concepts, network analysis

I. Introduction

Series compensation is found to be the most practical and economical method of increasing the power transmitting capability of long EHV transmission lines. Its application may, however, lead to self-excited oscillations caused by electrical subsynchronous resonance (SSR) as investigated by Rustebakke and Concordia. There are two types of self-excitation, one involving electromagnetic processes and the other involving electromagnetic and electromechanical processes in the system. The contributions of synchronous generator rotor action and of the induction generator effect to sustained subsynchronous oscillations are discussed by Bailleul and Goldberg. A state space model has been employed by Colin et al to represent the action of subsynchronous armature currents on turbine generator shaft torsional vibrations. A digital simulation program for the analysis of turbine generator shaft characteristics based on the state-space model is outlined. Colin et al have presented an eigenvalue analysis to study the three types of dynamic instability, namely, steady-state instability, electrical self-excitation, and torsional mode interactions that can be associated with the generator–turbine system connected to an infinite bus through a series-compensated transmission line. A rigorous mathematical technique, presented by Badr and El-Serafi, incorporates the synchronous machine governing and excitation control systems in the mathematical model, and considers the torsional dynamics for the evaluation of the subsynchronous resonance stability limit of a regulated synchronous machine. A computer algorithm based on the D-partition technique has been developed by Sreedharan and Ghodgaonkar to study the self-excited oscillations and hunting of a synchronous machine connected to series-compensated EHV lines of inherently low resistance. However, the mathematical model used by Sreedharan and Ghodgaonkar does not consider the torsional dynamics. In addition to the above limitation, the algorithm given in Reference 5 requires a large amount of computing time even to locate a single point on the stability limit curve in the $R-X$ plane. An improved formulation taking into consideration the torsional mode interaction is presented for the evaluation of the stability limit curves in the $R-X$ plane in Reference 6. The improved version is a modification of the algorithm presented in Reference 5. The modification

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Vol 13 No 5 October 1991 287
consists of applying the Newton method to obtain the critical value of the frequency on the border-line of the stability zones in an iterative manner. The modified approach results in a reduction of computing time due to the fact that the critical frequency is determined through a systematic iterative technique rather than through an exhaustive search as used in Reference 5.

In all previous studies on SSR, the parameters of the mechanical and electrical systems were assumed to be known exactly. However, inaccuracies in measurements, manufacturing tolerances and uncertainties affect the parameters of the mechanical and electrical systems, leading to deviations from their nominal values.

As a consequence, the stability boundary of the system in the $R-X_p$ plane will deviate from the calculated nominal stability boundary. Hence a probabilistic approach had been proposed for the first time\(^7\), where the accuracy of the stability limit curves in the $R-X_p$ plane have been determined for EHV lines with series compensation taking into consideration the nondeterministic nature of the generating system and transmission system parameters. Studies of Reference\(^7\) have shown that the effect of random system parameters is to increase the area of the unstable zone in the parametric plane. The main consequence of a probabilistic approach is that if the system is operated with the border-line value of $R$, obtained through a deterministic analysis, there is always a probability of self-excited oscillations existing in the system. In essence, it has been concluded\(^7\) that the potential modes of instability for a system with EHV lines should be determined based on a probabilistic model rather than on the basis of a deterministic model. It is worth mentioning that a similar approach has been used by Billinton and Kuruganty\(^4,9\) in the area of transient stability. Further it is necessary to point out here that probabilistic techniques are available and have been applied in many other areas of power systems.

The objective of the present paper is to investigate the effects of the probabilistic approach of analysis on the stability zones considering mechanical oscillations and mode interactions since these have not been considered in Reference\(^7\). First the mathematical models and the characteristic equations of electrical self-excitation, mechanical oscillation, and torsional interactions are developed. The nature of the random system parameters is then considered using the results of deterministic studies pertaining to the three phenomena in order to calculate the variances of the critical frequency, critical resonance, and the stability limit curves.

II. Mathematical models

II.1. Electrical self-excited oscillations

Self-excited electrical oscillations are due to the parametric resonance between the inductance and capacitance of a series-compensated transmission system. The natural frequency of such oscillations is always lower than the system frequency $w_o$. A balanced set of three phase armature currents at the resonant frequency $w_r$ of the electrical circuit produces a rotating magnetic field in the synchronous machine. As the rotor magnetic field which rotates at an angular frequency of $w_r$ overtake the more slowly rotating synchronous m.m.f. in the armature at $w_p$, the rotor resistance to the subsynchronous current viewed from the armature terminals of the machine becomes negative due to the induction generator effect. When this negative resistance exceeds the sum of the armature and network resistance at a resonant frequency, self-excitation prevails. The analysis of self-excitation phenomena is based on the characteristic equation of the system for small changes in the variables for a given incremental disturbing force. The small displacement model presented in Appendix I is based on Park's two-axis theory in accordance with IEEE notation\(^13\), and under the assumption of constant rotor speed equal to the synchronous speed during self-excitation.

The characteristic equation of the system shown in Figure 1 is obtained by equating the principal determinant of the system matrix equation (A.1.1) in Appendix I to zero and is given by

$$C_1(p)R^2 + C_2(p)R + C_3(p) = 0$$

where

$$C_1(p) = 1, \quad C_2(p) = 2A(p) + p(X_p + X_g + M(p)) - L(p)$$
$$C_3(p) = (A(p) + pX_g - L(p)) \times (A(p) + pX_g + pM(p))$$

$$L(p) = G(p)H_i(p) \left( \frac{V_{tdo}}{V_{to}} A(p) + \frac{V_{tdo}}{V_{to}} B(p) \right)$$
$$M(p) = G(p)H_i(p) \left( \frac{V_{tdo}}{V_{to}} A(p) - \frac{V_{tdo}}{V_{to}} B(p) \right)$$

For an unregulated synchronous machine $L(p) = M(p) = 0$.

11.2 Mechanical oscillations

Series compensation increases the effective value of the $(R/X)$ ratio of the transmission system. With this increase in the $(R/X)$ ratio, the machine damping torque is reduced, and hence there is a possibility of low-frequency self-excited oscillations (hunting) of the synchronous machine occurring. The study of electrical self-excitation\(^5\) has excluded consideration of hunting or electro-mechanical instability. Since electrical self-excitation and hunting appear simultaneously and influence each other, it is necessary to consider both while
investigating the operation of a synchronous generator feeding a series compensated transmission system.

The small signal model of the system as shown in Figure 1 considering the variation in the rotor speed of the synchronous machines due to hunting is given in Appendix 2. The characteristic equation is obtained by equating the principal determinant of the system matrix equation (A.2.1) of Appendix 2 to zero, and is given by

\[ C_1(p)R^2 + C_2(p)R + C_3(p) = 0 \]  

(2)

where

\[ C_1(p) = A_{33} \]
\[ C_2(p) = A_{12}A_{33} + A_{22}A_{33} - A_{23}A_{32} - A_{13}A_{32} \]
\[ C_3(p) = A_{11}(A_{22}A_{33} - A_{23}A_{32}) \]
\[ - A_{21}(A_{12}A_{33} - A_{13}A_{32}) \]
\[ + A_{31}(A_{12}A_{23} - A_{13}A_{22}) \]

II.3 Torsional mode interaction

In deriving the small displacement representation of the power system described in equation (A.2.1) of Appendix 2, the shaft and all the rotating masses coupled to it are assumed to oscillate as one block with an inertia constant \( H_q \). Since turbine-driven generators are capable of exhibiting torsional oscillations in modes corresponding to the various groups of rotating masses connected to the shaft, the interaction between these oscillatory modes and the self-excited oscillations of the synchronous machine in a power system can adversely affect the stability margin. It is necessary to modify the small displacement model of the power system to include the effect of shaft dynamics. The small displacement equations of the power system considering torsional dynamics in the form of a lumped parameter representation of the turbine-generator shaft assembly, as shown in Figure 2 are given in Appendix 3. The characteristic equation of the system is obtained by equating the determinant of the system matrix equation (A.3.1) of Appendix 3 to zero. The characteristic equation is written as

\[ C'_1(p)R^2 + C'_2(p)R + C'_3(p) = 0 \]  

(3)

where

\[ C'_1(p) = A'_{33} \]
\[ A'_{33} = A_{33} - A_{23}A_{32} \]
\[ C'_2(p) = A_{11}A'_{33} + A_{22}A'_{33} - A_{23}A_{32} - A_{13}A_{32} \]
\[ C'_3(p) = A_{11}(A_{22}A'_{33} - A_{23}A_{32}) \]
\[ - A_{21}(A_{12}A'_{33} - A_{13}A_{32}) \]
\[ + A_{31}(A_{12}A_{23} - A_{13}A_{22}) \]

![Figure 2. System under study](image)

III. Stability analysis in the \( R-X_e \) plane

III.1 Deterministic method of analysis

The method of analysis adopted in this paper is basically based on the frequency scanning method requiring the substitution of \( p = j\omega \) in the characteristic equation. Hence the general form of equations (1)–(3) for electrical self-excitation, mechanical oscillation, and mode interaction respectively is written as

\[ C_1(j\omega)R^2 + C_2(j\omega)R + C_3(j\omega) = 0 \]  

(4)

Two simultaneous equations are obtained by equating the real and imaginary parts of equation (4) to zero. Thus

\[ f_1(R, \omega, \eta_i, X_e) = C_{1R}R^2 + C_{2R}R + C_{3R} = 0 \]  

(5)

\[ f_2(R, \omega, \eta_i, X_e) = C_{1I}R^2 + C_{2I}R + C_{3I} = 0 \]  

(6)

where \( C_i = C_{iR} + jC_{iI}, i = 1, 2, 3 \) and \( C_{iR} \) and \( C_{iI} \) are functions of the frequency of oscillations \( \omega \), the system parameters \( \eta_i \), and the capacitive reactance \( X_e \).

For given values of \( \eta_i \) and \( X_e \), equations (5) and (6) are solved simultaneously by the Newton–Raphson method to yield \( \omega_{n_{i}} \) and \( R_{n_{i}} \) which are the critical frequencies at which the subsynchronous oscillations are set up, and the limiting resistance at which the oscillations can be damped out, respectively. The stability boundaries are then drawn in the \( R-X_e \) plane.

III.2 Probabilistic method of analysis

The random deviations of the mechanical and electrical system parameters from their nominal values are among the sources of uncertainty in an electric power system. These uncertainties are due to the inherent tolerances in the specified data, due to the nonavailability of accurate data for mechanical damping parameters, or to drifts caused by operating conditions. The system parameters can therefore be assumed to be random variables, and a probabilistic determination of the stability limit curve in the \( R-X_e \) plane becomes realistic.

In the probabilistic study, the system parameters \( \eta_i \) of the electrical and mechanical systems are assumed to be random variables with a normal distribution having a mean \( \mu_i \) equal to its nominal value. The standard deviation of the random variable is determined as follows

If the variation of the random variable from its mean \( \mu_i \) is ±10%, then the upper and lower limits (5) of the random variable are \( 1.1\mu_i \) and 0.9\( \mu_i \), respectively. If the random variable is assumed to be normally distributed, and can take any value between its upper and lower limits, then its standard deviation is given by

\[ 3\sigma_i = \frac{U - L}{2} = 0.1\mu_i \]

or

\[ \sigma_i = \frac{0.1}{3}\mu_i \]  

(7)

A deterministic study is first carried out with the given nominal system parameters to obtain the mean values \( R_{n_{i}} \) and \( \omega_{n_{i}} \) of \( R_{n_{i}} \) and \( \omega_{n_{i}} \), respectively. Having calculated \( R_{n_{i}} \) and \( \omega_{n_{i}} \), the variances \( \delta^2_{n_{i}} \) and \( \delta^2_{n_{i}} \) are obtained by following the procedure as outlined in Appendix 4, and making use of the variances of the system parameters calculated from equation (7). The probability density function and the distribution function are then...
evaluated for various values of $X_r$. The probabilistic stability limit curves in the $R-X_r$ plane are then obtained from the probability distribution function in the following manner.

First, a value for the probability of the system not exhibiting oscillations is selected. Then from the probability distribution function for a given $X_r$, the value of $R_{cr}$ corresponding to the above probability value is determined. Similarly, the value of $R_{cr}$ for all values of $X_r$ for the assumed probability value is obtained. The stability limit curve in the $R-X_r$ plane is drawn making use of the values of $R_{cr}$ and the corresponding value of $X_r$ for the assumed value of probability. Likewise, the stability limit curves in the $R-X_r$ plane for any probability value ranging from 0 to 1 are obtained.

IV. Numerical results and discussion
The deterministic and probabilistic methods of analysis described in this paper are applied to a system to determine its stability limit curves in the $R-X_r$ plane for

- electrical self-excitation
- mechanical oscillations
- torsional mode interaction

The system data is given in Appendix 5.

IV.1 Electrical self-excitation
The deterministic stability limit curves in the $R-X_r$ plane for an unregulated synchronous machine and a regulated machine with a proportional type voltage regulator for three different gains $U_p = 10, 50$ and 200 are determined.

These are shown in Figure 3. The probability density and distribution curves are shown in Figure 4 corresponding to the case of an unregulated machine for a value of $X_r = 0.5$ p.u., i.e. a series compensation of 44.5%. Similar curves are determined for all other values of $X_r$. These curves are then used to determine the probabilistic stability limit curves in the $R-X_r$ plane as shown in Figure 5, and as explained in Section III.2. It can be seen from Figure 5 that the unstable zone in the $R-X_r$ plane is extended if random variations of system parameters are taken into account. Figure 6 shows the stability limit curves with zero percent probability of the system not exhibiting self-excited oscillation for an unregulated synchronous machine with three different voltage regulator gains. It is observed that, as in the deterministic case, the conventional voltage regulator

![Figure 3. Stability limit curves for system with self-excited oscillations (deterministic)](image)

![Figure 4. Effect of system random parameters on limiting value of $R$, for $X_r = 0.5$ p.u.)](image)
IV.3 Torsional mode interaction

Figures 9 and 10 show the stability limit curves considering torsional interaction and deterministic system parameters. Figure 9 shows the deterministic limit curves with and without inclusion of the torsional dynamics in the representation of the system. In both cases the synchronous machine is assumed to be unregulated. Curve 'a' corresponds to the case of

![Figure 5: Probabilistic border-line stability curves](image1)

![Figure 6: Effect of voltage regulator gain on the stability limit curves with 0% probability of self-excited oscillations](image2)

gain does not increase the zone of instability, for practical values of $X_s$ in the 0–1 p.u. range, when the nondeterministic nature of the system parameters is taken into account.

IV.2 Mechanical oscillations

Figure 7 shows the deterministic stability limit curves considering electrical and mechanical oscillations. Curve 'a' shows the stability limit curves with electrical oscillation and curve 'b' shows the stability limit curves with mechanical oscillation. Curve 'c' shows a case of mechanical oscillations including the effect of regulator gain of value 10. Higher values of regulator gain have negligible effect on the stability limit curve for practical values of $X_s$. From Figure 7, it can be concluded that the effect of rotor motion should be included in the analysis of self-excited oscillation as its effect is to increase the unstable zone in the $R-X_s$ plane for low-frequency oscillations. It is further seen from Figure 7 that the voltage regulator of the type considered is not able to suppress the high-frequency oscillations.

![Figure 7: Stability limit curves with mechanical oscillations for the system shown in Figure 1 (deterministic)](image3)

Figure 8 shows the variation of system resistance $R$ for 0% probability of the system exhibiting mechanical oscillations for an unregulated machine and for a regulated machine with three different voltage regulator gains. It is observed that, as in the deterministic case, the conventional voltage regulator does not improve the stability limit even when the nondeterministic nature of the system parameters is considered.

![Figure 8: Effect of voltage regulator gain on the stability limit curves for 0% probability of mechanical oscillations](image4)

Vol 13 No 5 October 1991
V. Conclusions

This paper has dealt with the subsynchronous resonance phenomena in series-compensated EHV transmission lines and has considered the probabilistic nature of the system parameters in evaluating the stability limit curves in the $R-X_C$ plane. It is concluded that random system parameters should be taken into consideration, especially when mode interaction occurs, as in this case the probabilistic stability limit curve is widely separated from the deterministic stability limit curve. The zone of instability in general increases when random variations in system parameters are accounted for. Furthermore, it has been shown that an increase in the $d$-axis damper winding parameter values increases the stable zone of operation and vice versa. Finally, it is shown that the probabilistic stability limit curves are unaffected by the gain values of the conventional proportional type voltage regulator, a result which is similar to that observed in deterministic studies.

The system parameters considered for the probabilistic analysis are assumed to be statistically independent. The parameters are not statistically dependent in the equations. However, statistically independent system parameters may be handled if $E(X_i, X_j)$ are known for the dependent parameters as in Appendix 6.

In the above probabilistic studies the controller gain is not assumed to be a random variable. However, on repeating the above studies, considering the controller gain as a random variable, it has been verified that the random controller gain does not affect the results given in Figures 3, 7 and 8, respectively. In this work, simple conventional voltage regulator is considered to see the effect of voltage gain on the stability limit.

References


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Appendix 1: mathematical model for electrical self-excited oscillations

The small-signal model of a regulated synchronous machine connected to a larger power system through a series-compensated transmission line as shown in Figure 1 may be written in the following form

\[
\begin{bmatrix}
R + a_{11} & a_{12} \\
 a_{21} & R + a_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta i_d \\
\Delta i_q
\end{bmatrix} = 0
\]  

(A1.1)

where

\[
a_{11} = A(p) + pX_d(p) \\
+ G(p)H_1(p) \left( \frac{V_{tqo}}{V_{to}} A(p) + \frac{V_{tdo}}{V_{to}} B(p) \right)
\]

\[
a_{12} = -(B(p) + X_d(p)) \\
+ G(p)H_1(p) \left( \frac{V_{tdo}}{V_{to}} A(p) - \frac{V_{tqo}}{V_{to}} B(p) \right)
\]

\[
a_{21} = B(p) + X_q(p) \\
+ pG(p)H_1(p) \left( \frac{V_{tqo}}{V_{to}} A(p) + \frac{V_{tdo}}{V_{to}} B(p) \right)
\]

\[
a_{22} = A(p) + pX_q(p) \\
- pG(p)H_1(p) \left( \frac{V_{tdo}}{V_{to}} A(p) - \frac{V_{tqo}}{V_{to}} B(p) \right)
\]

\[V_{tdo}, V_{tqo}\] direct and quadrature axis components of machine terminal voltage under initial state of the system

\[V_{to}\] machine terminal voltage under initial state

\[A(p)\] quadrature-axis transmission system operational reactance

\[B(p)\] direct-axis transmission system operational reactance

\[X_d(p)\] quadrature-axis operational impedance of generator

\[X_q(p)\] direct-axis operational impedance of generator

\[G(p)\] operational impedance of the excitation system

\[H_1(p)\] transfer function of the excitation controller

\[R\] combined resistance of generator, line, and transformers

The expressions used for the operational impedance of the machine and the transmission line and for the transfer function of the excitation controller are given below.

\[X_d(p) = X_d \left( \frac{p^2 X_{qad} - 2 X_{qd} + X_{fqd}}{p^2 X_{kad} X_{qad} - X_{kad}^2} + \frac{p X_{d2} (R_{ad} + R_{fd})}{p^2 X_{kad} X_{qad} - X_{kad}^2 + p (X_{qad} R_{ad} + X_{fad} R_{kad}) + R_{kad} R_{fd}} \right) \]

(A1.2)

\[X_q(p) = X_q \left( \frac{p^2 X_{d2} - (p X_{qad} + R_{qad})}{p^2 X_{kad} X_{qad} - X_{kad}^2} \right) \]

(A1.3)

\[G(p) = \frac{p^2 X_{kad} X_{qad} - X_{kad}^2 + p (X_{qad} R_{ad} + X_{fad} R_{kad})}{p^2 X_{kad} X_{qad} - X_{kad}^2 + p (X_{qad} R_{ad} + X_{fad} R_{kad}) + R_{kad} R_{fd}} \]

(A1.4)

\[A(p) = X_d p + \frac{X_d}{p^2 + 1} \]

(A1.5)

\[B(p) = X_q(p) - \frac{X_q}{p^2 + 1} \]

(A1.6)

\[H_1(p) = \frac{U_e}{U_e + U_{t} + (1 + T_s) U_{t} (1 + T_s) U_{t}} \]

(A1.7)

The initial values of \(V_{td}, V_{tq}\) and \(V_{t}\), viz. \(V_{do}, V_{tqo}\) and \(V_{to}\) are computed from the initial load on the system and the amount of series compensation. Referring to Figure A1, the relevant equations can be written as

\[I_R = \frac{S_R}{V_R} = \frac{P_R - jQ_R}{V_R} \]

(A1.8)

\[X_r = X_d - X_q \]

(A1.9)

\[V_r = V_R + jI_R X_r \]

(A1.10)
\[ E_q = V_i + jI_k X_q \]  
\[ \phi = \tan^{-1} \left( \frac{\text{Im}(E_q)}{\text{Re}(E_q)} \right) \]  
\[ l_{q0} = |I_k| \cos(\phi - \delta) \]  
\[ l_{d0} = |I_k| \sin(\phi - \delta) \]  
\[ \theta = \tan^{-1} \left( \frac{\text{Im}(V_i)}{\text{Re}(V_i)} \right) \]  
\[ V_{wo} = V_i \cos(\theta - \delta) \]  
\[ V_{wq} = -V_i \sin(\theta - \delta) \]  
\[ V_{do} = -|V_i| \sin \delta \]  
\[ V_{dq} = |V_i| \cos \delta \]  
\[ \psi_{do} = E_d + i_{d0}(X_d - X_q) + i_{d0} X_d \]  
\[ \psi_{dq} = i_{q0} X_q \]  

where

- \text{Im} \quad \text{imaginary part of}
- \text{Re} \quad \text{real part of}
- l_{d0}, l_{q0} \quad d, q \text{ axis component of armature current, respectively}
- P_B \quad \text{active power delivered to infinite bus}
- Q_B \quad \text{reactive power delivered to infinite bus}
- p \quad \text{incremental susceptance}
- R_{fd}, R_{fq} \quad d, q \text{ axis field winding resistance}
- R_{d0}, R_{q0} \quad d, q \text{ axis damper winding resistance, respectively}
- V_{d1}, V_{q1} \quad d- \text{ and q-axis component of } V_1, \text{ respectively}
- V_{d0}, V_{q0} \quad d- \text{ and q-axis component of } V_0, \text{ respectively}
- E_q \quad \text{generator voltage behind synchronous reactance in q-axis}
- X_e \quad \text{external reactance}
- X_t \quad \text{fine reactance}
- X_s \quad \text{series compensation reactance}
- X_{d1}, X_{q1} \quad d- \text{ and q-axis synchronous reactances, respectively}
- X_{do}, X_{qo} \quad d- \text{ and q-axis initial value of flex linkage reactances}
- X_{d}(p), X_{q}(p) \quad d \text{ and q-axis operating impedances, respectively}
- X_{d1}, X_{q1} \quad d \text{ and q-axis magnetizing reactances, respectively}
- X_{r}, X_{q} \quad d \text{ axis field winding self reactance}
- X_{r1}, X_{q1} \quad d \text{ and q-axis damper winding self reactance, respectively}
- T_c \quad \text{measuring device time constant}
- T_s \quad \text{amplifier time constant}
- T_0 \quad \text{exciter time constant}
- T_{sc} \quad \text{feedback stabilizing loop time constant}
- U_a \quad \text{amplifier gain}
- U_s \quad \text{measuring device gain}
- U_{sc} \quad \text{feedback stabilizing loop gain}
- \delta \quad \text{torque angle}
- \Delta \quad \text{prefix to denote small changes about initial operating point}
- \theta_0 \quad \text{suffix to denote initial operating conditions}

### Appendix 2: Mathematical model for mechanical oscillations

The small signal model is of the following form

\[
\begin{bmatrix}
R + A_{11} & A_{12} & A_{13} & \Delta l_{d1} \\
A_{14} & R + A_{22} & A_{23} & \Delta l_{q2} \\
A_{15} & A_{12} & A_{33} & \Delta l_{d1} \\
A_{16} & A_{13} & A_{33} & \Delta l_{q2}
\end{bmatrix} = 0
\]  

where

- \[ A_{11} = A(p) + p X_d(p) - D_{14} \]
- \[ A_{12} = -B(p) - D_{12} \]
- \[ A_{13} = D_1 - D_{13} - \chi_{do} P \]
- \[ A_{21} = B(p) + X_d(p) + p D_{14} \]
- \[ A_{22} = A(p) + p D_{12} \]
- \[ A_{23} = \chi_{dq} P + D_2 + p D_{13} \]
- \[ A_{31} = \chi_{do} + i_{q0} X_d + i_{d0} D_{14} \]
- \[ A_{32} = -\chi_{dq} + i_{q0} D_{13} \]
- \[ A_{33} = H g p^2 + i_{q0} D_{13} \]
- \[ D_1 = V_{d0} + i_{q0} A_{14}(p) + i_{q0} B(p) \]
- \[ D_2 = -V_{d0} + i_{q0} B(p) - i_{q0} A_{14}(p) \]
- \[ D_{11} = -G(p) H_{11}(p) \left( \frac{V_{d0}}{V_{d0}} + \frac{V_{q0}}{V_{q0}} A(p) \right) \]
- \[ D_{12} = -G(p) H_{12}(p) \left( \frac{V_{d0}}{V_{d0}} A(p) + \frac{V_{q0}}{V_{q0}} B(p) \right) \]
- \[ D_{13} = -G(p) H_{13}(p) \left( \frac{V_{d0}}{V_{d0}} D_2 + \frac{V_{q0}}{V_{q0}} D_1 \right) \]

\( H g \) is the total inertia constant of the rotating masses.

For an unregulated synchronous machine

\[ D_{11} = D_{12} = D_{13} = 0 \]

### Appendix 3: Mathematical model for torsional interaction

The mathematical model can be written as

\[
\begin{bmatrix}
R + A_{11} & A_{12} & A_{13} & 0 & \Delta l_{d1} \\
A_{14} & R + A_{22} & A_{23} & 0 & \Delta l_{q2} \\
A_{15} & A_{12} & A_{33} & 0 & \Delta l_{d1} \\
0 & 0 & 0 & A_{44} & \Delta T_f
\end{bmatrix} = 0
\]  

where \( A_{11}, \ldots, A_{33} \) are defined in Appendix 2, and

- \[ A_{13} = H g p^2 + (D_{q0} + D_{T_{1q}}) p + K_{T_{1q}} \]
- \[ A_{14} = A_{44} - (D_{T_{1q}} + K_{T_{1q}}) \]
- \[ A_{44} = H T_1 p^2 + (D_{T_{1q}} + D_{T_{1q}}) p + K_{T_{1q}} \]

where

- \( H g, H T_1 \) \text{ turbine and generator inertia constants, respectively}
- \( D_{q0}, D_{T_{1q}} \) \text{ turbine and generator load damping, respectively}
- \( D_{T_{1q}} \) \text{ turbine-generator torsional damping}
- \( K_{T_{1q}} \) \text{ turbine-generator shaft stiffness}
- \( T_i \) \text{ electromagnetic torque}
- \( \delta_g \) \text{ generator rotor angle}
Appendix 4: determination of $\sigma^2_{w_{cri}}$ and $\sigma^2_{\Delta w_{cri}}$

4.1 Determination of $\sigma^2_{w_{cri}}$

Eliminating $R^2$ from the real and imaginary parts of the characteristic equations (5) and (6) given in Section III.1 and solving for $R$, we obtain

$$R_{cri} = F(w_{cri}, n_0^0, X_c) \quad \quad (A4.1)$$

Substituting equation (A4.1) into equation (5), we obtain

$$f(w_{cri}, n_0^0, X_c) = 0 \quad \quad (A4.2)$$

The Taylor series expansion of this equation, neglecting higher-order derivatives, yields

$$\Delta f = \frac{\partial f}{\partial w_{cri}}|_{w_{cri}^0, n_0^0} \Delta w_{cri} + \sum_{i=1}^{n} \frac{\partial f}{\partial n_i} \bigg|_{w_{cri}^0, n_0^0} \Delta n_i = 0 \quad \quad (A4.3)$$

or

$$\Delta w_{cri} = \left( \sum_{i=1}^{n} \frac{\partial f}{\partial n_i} \bigg|_{w_{cri}^0, n_0^0} \right)^{-1} \frac{\partial f}{\partial w_{cri}} \bigg|_{w_{cri}^0, n_0^0} \Delta n_0 \quad \quad (A4.4)$$

The nominal value of the critical frequency $w_{cri}$ is calculated from a deterministic study corresponding to the nominal system parameters $n_0^0$ and is taken to be the mean value of $w_{cri}$. Assuming statistical independence between the system parameters, we obtain

$$\sigma^2_{w_{cri}} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial n_i} \bigg|_{w_{cri}^0, n_0^0} \right)^2 \sigma^2_i \quad \quad (A4.5)$$

Thus, the variance of $w_{cri}$ for assumed variances of the system parameters is obtained. The mean $w_{cri}^0$ and the variance $\sigma^2_{w_{cri}}$ given by equation (A4.5) completely describe the probability distribution of $w_{cri}$.

4.2 Determination of $\sigma^2_{\Delta w_{cri}}$

The Taylor series expansion of equation (A4.1) around the mean value of $w_{cri}$ and $n_0$ and neglecting the second- and higher-order derivatives yields

$$\Delta R_{cri} = \frac{\partial F}{\partial w_{cri}}|_{w_{cri}^0, n_0^0} \Delta w_{cri} + \sum_{i=1}^{n} \frac{\partial F}{\partial n_i} \bigg|_{w_{cri}^0, n_0^0} \Delta n_i \quad \quad (A4.6)$$

Therefore

$$\sigma^2_{\Delta w_{cri}} = \left( \frac{\partial F}{\partial w_{cri}} \bigg|_{w_{cri}^0, n_0^0} \right)^2 \sigma^2_{w_{cri}} + \sum_{i=1}^{n} \left( \frac{\partial F}{\partial n_i} \bigg|_{w_{cri}^0, n_0^0} \right)^2 \sigma^2_i \quad \quad (A4.7)$$

The mean value of $R_{cri}$ viz. $R_{cri}^0$ is obtained from a deterministic study with nominal system parameters $n_0^0$, $R_{cri}^0$, and $\sigma^2_{\Delta w_{cri}}$ completely characterize the probability distribution of $R$.

Appendix 5: system data

Basic MVA = 1000
Base $KV = 500$

| $X_c$ | 1.1224 p.u. | $X_{fcd}$ | 1.0970 p.u. |
| $R$ | 0.0050 p.u. | $R_{fcd}$ | 0.0004 p.u. |
| $X_d$ | 1.0000 p.u. | $X_{fcd}$ | 1.0070 p.u. |
| $X_q$ | 0.6600 p.u. | $R_{sd}$ | 0.0270 p.u. |
| $X_{sd}$ | 0.8470 p.u. | $X_{sd}$ | 0.7670 p.u. |
| $X_{si}$ | 0.5070 p.u. | $R_{si}$ | 0.0340 p.u. |
| $H_{T_1}$ | 935.90 p.u. | $H_{T_2}$ | 1559.0 p.u. |
| $D_{sw}$ | 0.0000 p.u. | $U_i$ | 0.0500 p.u. |
| $D_{T_{1f}}$ | 0.6000 p.u. | $U_e$ | 1.0000 p.u. |
| $D_{T_{1g}}$ | 0.6000 p.u. | $T_{s}$ | 7.5400 p.u. |
| $K_{f_{1g}}$ | 22.5000 p.u. | $T_{n}$ | 0.0000 p.u. |
| $K_{e}$ | 1.0000 p.u. | $T_{s}$ | 301.600 p.u. |
| $T_{j}$ | 377.000 p.u. |

Appendix 6: dependent random variables

If the random variables are not independent and the $E(X_i, X_j)$ are known, then the mean variance of the statistically dependent variables can be handled as below. Let the function $Y$ of the dependent variables $X_1, X_2, \ldots, X_n$ be given as

$$Y = g(X_1, X_2, \ldots, X_n)$$

If $Y$ is represented by the first-order Taylor series expansion about the point $\mu_1, \mu_2, \ldots, \mu_n$

$$Y = g(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(\mu_1, \mu_2, \ldots, \mu_n)(X_i - \mu_i)$$

i.e.

$$Y = a_0 + \sum_{i=1}^{n} a_i (X_i - \mu_i)$$

where

$$a_0 = \mu_y = E(Y) = g(\mu_1, \mu_2, \ldots, \mu_n)$$

$$a_i = \frac{\partial g}{\partial X_i}(\mu_1, \mu_2, \ldots, \mu_n)$$

$$E(Y - \mu_y)^2 = E \left( \sum_{i=1}^{n} a_i (X_i - \mu_i)^2 \right)$$

i.e.

$$\sigma^2 \approx \sum_{i=1}^{n} a_i^2 \sigma X_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}$$

i.e.

$$\sigma_{ij} = E((X_i - \mu_i)(X_j - \mu_j)) = E(X_i, X_j) - \mu_i \mu_j$$

Hence if $E(X_i, X_j)$ is known for the dependent variables then the combined mean $\mu_Y$ and variance $\sigma^2_Y$ can be found.