EXTREMAL PROPERTIES AND COEFFICIENT ESTIMATES
FOR POLYNOMIALS WITH RESTRICTED ZEROS AND ON
LOCATION OF ZEROS OF POLYNOMIALS

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(K.K. Dewan)

(KUM KUM DEWAN)
This thesis consists of three chapters. The first chapter deals with the extremal properties of a polynomial. In the second chapter we have studied the problems concerning the location of zeros of a polynomial and in the third chapter some coefficient problems for polynomials have been considered.

If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then according to Bernstein's theorem

\[
\max_{|z|=1} \left| p'(z) \right| \leq n \max_{|z|=1} |p(z)|, \tag{1}
\]

\[
\max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|, \tag{2}
\]

\[
\int_{0}^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leq n^2 \int_{0}^{2\pi} |p(e^{i\theta})|^2 \, d\theta \tag{3}
\]

and

\[
\int_{0}^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leq R^{2n} \int_{0}^{2\pi} |p(e^{i\theta})|^2 \, d\theta \tag{4}
\]

Lax [48] and Ankeny and Rivlin [3] considered the class of polynomials \( p(z) \) having no zeros inside the unit circle and obtained inequalities analogous to (1) and (2). Rahman [67] considered the class of polynomials having no zero in \( |z| < K, K \leq 1 \) and obtained the inequalities analogous to (3) and (4). The class of polynomials having no zero in \( |z| < K, K \geq 1 \) was studied by Malik [49]. (Also see Govil...
and Rahman [35]). Govil, Jain and Labelle [33] considered the class of polynomials satisfying \( p(z) = z^n p\left(\frac{1}{z}\right) \) and having the zeros either in the left half plane or in the right half plane while O'hara and Rodriguez [60] considered the class of polynomials satisfying \( p(z) = z^n p\left(\frac{1}{z}\right) \). In the first chapter we consider the class of polynomials satisfying \( p(z) = z^n p\left(\frac{1}{z}\right) \) and prove a result of Govil, Jain and Labelle [33] without the condition that \( p(z) \) has all its zeros in one half plane. Our result is best possible and generalizes the result due to Govil, Jain and Labelle [33]. Also we obtain \( L^2 \) inequalities for the \( s \)th derivative of the classes of polynomials having no zero in \(|z| < K, K \leq 1\) and for polynomials having no zero in \(|z| < K, K \geq 1\). These results generalize some well known results. The inequality for the case \( K \leq 1 \), is best possible. For the class of polynomials having no zeros in \(|z| < K, K \leq 1\) we as well obtain inequality analogous to (2). Besides some other results for these classes and other related classes have also been obtained.

A classical result of Cauchy on the location of the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) states that all the zeros are in the circle \(|z| \leq 1+A\) where

\[
A = \max_{0 \leq j < n} |a_j|.
\]

Joyal, Labelle and Rahman [42] and
Brham Datt and Govil [18] improved the above mentioned result of Cauchy. Further various generalizations of Eneström-Kakeya Theorem have been obtained, among others by Cargo and Shisha [3a], Joyal, Labelle and Rahman [42], P.V. Krishnaiah [46], Rubinstein [75], Govil and Rahman [34] and Govil and Jain [32]. In the second chapter we obtain an improvement of the result of Joyal, Labelle and Rahman [42] and Brham Datt and Govil [18] dealing with Cauchy's theorem by obtaining the zero free region bigger than the zero free region obtained by them. We also sharpen the result proved by Joyal, Labelle and Rahman [42] and Govil and Jain [32] concerning Eneström-Kakeya theorem by obtaining a smaller region. Besides these, various other results concerning the location of zeros of polynomials have been obtained.

Let \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree \( n \).

If \( M = \max_{|z|=1} |p(z)| \), then, by Cauchy's inequality, we have

\[
|a_{\nu}| \leq M, \quad 0 \leq \nu \leq n. \tag{5}
\]

Also, by a very simple method C. Visser [86] proved that

\[
|a_{0}| + |a_{n}| \leq M \tag{6}
\]

Rahman [66] gave the estimate for the sum of the moduli of any two coefficients of a polynomial \( p(z) \) in terms of \( M = \max_{|z|=1} |p(z)| \) and proved that for \( 0 \leq u < v \leq n \)

O'hara and Rodriguez \cite{6o} considered the class of polynomials having all its zeros on the unit circle. The class of polynomials \( p(z) = \sum_{v=0}^{n} a_v z^v \) having a zero on \(|z| = \rho\), \((0 < \rho < \infty)\) was studied by Rahman and Schmeisser \cite{70} and they obtained a sharp upper bound for \(|a_0|\) in terms of \(\max_{|z|=1} |p(z)|\). In the third chapter we consider the class of polynomials having a prescribed zero on \(|z| = 1\) and obtain an inequality analogous to (5). We also obtain inequalities analogous to (7) for the class of polynomials having all its zeros on \(|z| = 1\) and for the class of polynomials whose zeros lie on or exterior (interior) to \(|z| = 1\). Besides these, some other related results have also been obtained.

\[ |a_u| + |a_v| \leq \frac{4M}{\pi} \] (7)
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