Short Communication

A note on pseudo-invexity and symmetric duality

S. Chandra *, V. Kumar

Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi 110016, India

Received 27 September 1994: accepted 24 October 1996

Abstract

A pair of symmetric dual nonlinear programming problems is presented and duality theorems are established under pseudo-invexity type assumptions on the kernel function. This formulation removes certain inconsistencies in a recently introduced primal–dual pair and gives the correct proof of various duality theorems.

Keywords: Symmetric duality; Nonlinear programming; Fractional programming; Pseudoinvexity

1. Introduction

Following the earlier work of Dom [5], Dantzig et al. [4] and Mond [8] on symmetric duality, many researchers attempted to generalize the formulation and weaken the convexity–concavity hypothesis required on the kernel function $K(x, y)$. Mond and Weir [10] weakened the convexity–concavity hypothesis for $K(x, y)$ to pseudoconvexity–pseudconcavity and Chandra et al. [3] studied symmetric duality in fractional programming. In [2], Chandra et al. studied symmetric duality under pseudo-invexity and gave an interpretation of the main construction in terms of a constrained two person game.

Recently, Nanda and Das [11] attempted to construct a pair of symmetric dual programming problems under the pseudo-invexity type restrictions on the kernel function and derive a symmetric duality theorem in nonlinear fractional programming. The purpose of this note is two fold. First, to point out that the construction of Nanda and Das [11] is not correct, and second to emphasize again (as done already in [6]) that for studying symmetric duality under pseudoconvexity or pseudo-invexity type assumptions, the construction of the dual pair has to be on the lines of Mond and Weir [10] and not on the lines of Dantzig et al. [4].

Here it is remarked that the nature of the mistake in the construction of Nanda and Das [11] is similar to that of Mishra et al. [7], which has been corrected and modified by Kumar et al. [6].

Certain other assumptions made by Nanda and Das [11] which do not seem to be valid, are also pointed out at appropriate places. We follow the notations of Nanda and Das [11] and present our results very briefly.

2. Problem formulation and prerequisites

Let $R^n$ denote the $n$ dimensional Euclidean space and $R^n_+$ be its nonnegative orthant. We have used the following definitions in the sequel.
Definition 1. A convex set $C$ of $\mathbb{R}^n$ is called a convex cone if for each $x \in C$ and $\lambda \geq 0$, $\lambda x \in C$. 

Definition 2. $C^\diamond = \{z \in \mathbb{R}^n : x^T z \leq 0 \text{ for all } x \in C\}$ is called the polar of the cone $C$. 

Definition 3. Let $S \subseteq \mathbb{R}^n$ be open and $f : S \to \mathbb{R}$. The function $f$ is said to be pseudoinvex with respect to $\eta$ on $S$, where $\eta$ is a function from $S \times S$ to $\mathbb{R}^n$, if 

$$[\eta(x,u)]^T \nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u) \quad \text{for all } x,u \in S.$$ 

Let $C_1, C_2$ be closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $S_1 \subseteq \mathbb{R}^n$ and $S_2 \subseteq \mathbb{R}^m$ be open and $T = S_1 \times S_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$. Let $C_1 \times C_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and $p : T \to \mathbb{R}$, be a twice differentiable function which is pseudoinvex in the first variable with respect to $\eta$ and $-p$ is pseudoinvex in the second variable with respect to $\eta$. Note that, $n_1 : S_1 \times S_1 \to \mathbb{R}^n$ and $n_2 : S_2 \times S_2 \to \mathbb{R}^m$. Further let $\nabla_x p(x,y)$ and $\nabla_y p(x,y)$ be the first and second order gradient vectors with respect to the first variable. $\nabla_y p(x,y)$, $\nabla_{yy} p(x,y)$, $\nabla_x p(x,y)$ and $\nabla_{xy} p(x,y)$ are defined similarly.

We present the following pair of symmetric dual programming problem. 

\begin{align*}
(P) \quad \text{Min } & p(x,y) \\
\text{subject to } & \nabla_y p(x,y) \in C_2^*, \\
& y^T \nabla_x p(x,y) \geq 0,
\end{align*}

\begin{align}
x \in C_1, \quad & \tag{1}
\end{align}

\begin{align*}
(D) \quad \text{Max } & p(u,v) \\
\text{subject to } & -\nabla_x p(u,v) \in C_1^*,
\end{align*}

\begin{align}
u^T \nabla_y p(u,v) \leq 0, \quad & \tag{4}
v \in C_2. \quad & \tag{5}
\end{align}

We shall discuss the symmetric duality for the pair (P) and (D) under the following assumptions similar to [9]:

$$\eta_i(x,u) + u \in C_1, \quad \text{for all } (x,u) \in C_1 \tag{6}$$

$$\eta_2(u,v) + y \in C_2, \quad \text{for all } v, y \in C_2. \tag{7}$$

It may be noted that this assumption has not been taken by Nanda and Das in Theorem 1 of [11] while in Theorem 2 it has been taken but with the condition that $\eta_i (i = 1, 2)$ is a function from $C_1 \times C_2$ to $C_i$. If, in particular, $\eta_i(x,u) = (x - u)$ and $C_i = \mathbb{R}^n_+$, then it amounts to saying that $x \geq 0, u \geq 0$ implies $x - u \geq 0$, which is not true. Also the proof of Theorem 2 in [11] is not correct because of improper application of the Fritz--John Theorem. 

On the other hand, let us take $p(x,y) = e^{x-y}$. It can be seen that $p(x,y)$ is pseudoconvex in $x$ for a fixed $y$ and pseudocoercive in $y$ for a fixed $x$. Hence satisfies the pseudoinvexity conditions of Nanda and Das [11] with $\eta(x,u) = (x - u)$. With this $p(x,y)$, the primal and dual problems of [11] reduce to the following

\begin{align*}
(P) \quad & \text{Min } f(x,y) = e^{x-y} + ye^{x-y} \\
\text{subject to } & -e^{x-y} \in C_2^*,
\end{align*}

\begin{align*}
(x,y) \in C_1 \times C_2. \tag{10}
\end{align*}

\begin{align*}
(D) \quad & \text{Max } g(x,y) = e^{x-y} + xe^{x-y} \\
\text{subject to } & -e^{x-y} \in C_1^*, \\
(x,y) \in C_1 \times C_2. \tag{11}
\end{align*}

Taking $C_1 = \mathbb{R}^+, C_2 = \mathbb{R}^*$, problems (P) and (D) become

\begin{align*}
(P) \quad & \text{Min } e^{(x-y)}(1 + y), \text{ s.t. } x, y \geq 0,
\end{align*}

\begin{align*}
(D) \quad & \text{Max } e^{(x-y)}(1 - x), \text{ s.t. } x, y \geq 0.
\end{align*}

It is easily seen that $f(0,1) = 2/e < 1 = g(0,0)$, and hence the weak duality theorem is contradicted. 

In the next section, we give duality theorems for the pair (P) and (D).

3. Duality theorem

**Theorem 1. (Weak duality.)** Let $(x,y)$ be feasible for (P) and $(u,v)$ be feasible for (D). Then, $\inf(P) \geq \sup(D)$. 

**Proof.** By (4) and (7),

$$- (\eta_i(x,u) + u)^T \nabla_x p(u,v) \leq 0,$$
(\eta_1(x,u) + u)\nabla_s p(u,v) \geq 0.

Also

\(-u)\nabla_s p(u,v) \geq 0.

Thus we have, \eta_1(x,u)\nabla_s p(u,v) \geq 0, which because of the pseudoinvexity of \( p(x,y) \) in the first variable with respect to \( \eta_1 \) implies,

\[ p(x,v) \geq p(u,v). \]

Similarly by (1) and (7) we obtain

\[-(\eta_2(v,y))\nabla_s p(x,y) \geq 0, \]

which because of the pseudoinvexity of \(-p(x,y)\) in the second variable with respect to \( \eta_2 \) implies,

\[ p(x, y) \geq p(x, v). \]

Therefore

\[ p(x, y) \geq p(u, v), \]

and hence the theorem follows. \( \square \)

**Theorem 2.** (Strong duality) Let \((\bar{x}, \bar{y})\) be optimal to \((P)\). Let the matrix \(\nabla_s p(\bar{x}, \bar{y})\) be nonsingular and \(\nabla_s p(\bar{x}, \bar{y}) \neq 0\). Then \((\bar{x}, \bar{y})\) is an optimal solution to the dual problem.

**Proof.** Since \((\bar{x}, \bar{y})\) is optimal for \((P)\), by the Fritz–John conditions given by Bazaraa and Goode [1] for symmetric dual nonlinear programming problems defined on convex cone domain, \(\exists \lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{C}_2, \lambda_1 \in \mathbb{R}\) such that the following are satisfied:

\[ \left[ \lambda_1 \nabla_s p(\bar{x}, \bar{y}) - \nabla_s p(\bar{x}, \bar{y})(\lambda_2 \bar{y} - \lambda_2) \right](x - \bar{x}) \geq 0, \]

for all \(x \in C_1, \) \(\lambda_1, \lambda_2 \neq 0\). \(\lambda_1 \in \mathbb{R}\) \(\lambda_2 \in \mathbb{C}_2, \lambda_1 \neq 0\) such that the following are satisfied:

\[ \lambda_1 \nabla_s p(\bar{x}, \bar{y}) - \nabla_s p(\bar{x}, \bar{y})(\lambda_2 \bar{y} - \lambda_2) = 0, \]

\[ \lambda_1 \lambda_2 \lambda_3 \neq 0. \]

\[ \lambda_1 \geq 0, \lambda_2 \in \mathbb{C}_2. \lambda_3 \geq 0. \]

Multiplying (9) by \((\lambda_2 - \lambda_2 \bar{y})\) and using (10) and (11) we have,

\[ (\lambda_2 - \lambda_2 \bar{y}) \nabla_s p(\bar{x}, \bar{y})(\lambda_2 - \lambda_2 \bar{y}) = 0. \]

Since it is assumed that \(\nabla_{ss} p(\bar{x}, \bar{y})\) is nonsingular, it follows that

\[ \lambda_2 = \lambda_2 \bar{y}. \]

Thus, from (9) it follows that \((\lambda_1 - \lambda_2)\nabla_s p(\bar{x}, \bar{y}) = 0\) and since, by assumption, \(\nabla_s p(\bar{x}, \bar{y}) \neq 0\) we have,

\[ \lambda_1 = \lambda_2. \]

If \(\lambda_1 = 0\), then \(\lambda_1 = 0\) and by (15) \(\lambda_2 = 0\), contradicting (12). Thus \(\lambda_1 > 0\) and hence \(\lambda_2 > 0\). Thus, by (15) we have \(\bar{y} \in C_2\).

Further by (13) and (8), for all \(x \in C_1\),

\[ \nabla_s p(\bar{x}, \bar{y})(x - \bar{x}) \geq 0. \]

Let \(x \in C_1\), then \(\bar{x} + x \in C_1\), and so the inequality (16) implies that \(\nabla_s p(\bar{x}, \bar{y})(x) \geq 0\) for every \(x \in C_1\), i.e., \(-\nabla_s p(\bar{x}, \bar{y}) \in C_1\). Also, by letting \(x = 0\) and \(x = 2\bar{x}\) in the inequality (16), simultaneously, we get \(\nabla_s p(\bar{x}, \bar{y}) = 0\). Thus, \((\bar{x}, \bar{y})\) is feasible for \((\bar{D})\) and the value of the objective function of \((P)\) and \((\bar{D})\) is the same at \((\bar{x}, \bar{y})\). Optimality follows by the weak duality theorem. \( \square \)

4. Special case

(1) Let \(p(x,y) = \phi(x,y)/(\psi(x,y))\) where \(\phi\) and \(\psi\) are real valued functions on \(C_1 \times C_2\), such that, \(\phi(\cdot, y)\) and \(\psi(x, \cdot)\) are convex, \(\phi(x, \cdot)\) and \(\psi(\cdot, y)\) are concave. Further it is assumed that \(\psi > 0\) and \(\phi > 0\). We may observe that \((\phi(x,y))/(\psi(x,y))\) is pseudo-convex in \(x\) for a fixed \(y\) and pseudo-concave in \(y\) for a fixed \(x\) and hence satisfies required pseudoinvexity assumptions with \(\eta_1(x,u) = (x-u)\) for arbitrary but fixed \(y\) and \(\eta_2(y,u) = (y-u)\) for arbitrary but fixed \(x\). Hence for \(p(x,y)\) defined as such the pair \((P)\) and \((\bar{D})\) is equivalent to the following

\[ \text{(P)} \quad \text{Min} \quad \frac{\phi(x,y)}{\psi(x,y)} \]

subject to

\[ \psi(x,y)\nabla_s \phi(x,y) - \phi(x,y)\nabla_s \psi(x,y) \in C_2. \]

\[ y' \left[ \psi(x,y)\nabla_s \phi(x,y) - \phi(x,y)\nabla_s \psi(x,y) \right] \geq 0, \]

\[ x \in C_1. \]

\[ \text{(\bar{D})} \quad \text{Max} \quad \frac{\phi(u,v)}{\psi(u,v)} \]
subject to
\[-\psi(u,v)\nabla_x \phi(u,v) - \phi(u,v) \nabla_x \psi(u,v) \in C^*_1,\]
\[u^T[\psi(u,v)\nabla_x \phi(u,v) - \phi(u,v)\nabla_x \psi(u,v)] \leq 0,\]
\[v \in C^*_2,\]
which extends the symmetric duality results of Chandra et al. [3] to convex cone domains.

Acknowledgements

The authors wish to thank the referees for several valuable suggestions which have considerably improved the presentation of this paper.

References