FINITELY PRESENTED GROUPS AND UNITS IN GROUP RINGS

SWATI MAHESHWARI

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

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SWATI MAHESHWARI

Department of Mathematics

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **FINITELY PRSENTED GROUPS AND UNITS IN GROUP RINGS** submitted by **Ms. Swati Maheshwari** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi

May, 2017

Prof. R. K. Sharma
Department of Mathematics
Indian Institute of Technology Delhi

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Abstract

This thesis is a study of unit groups of rings and group rings. We concentrate on generators and presentations of General linear groups. For every $n > 1$, we find a set of generators of $GL(2, \mathbb{Z}_n)$ in terms of Lie regular units, which is a generalization of the work done by Sharma et al.. Further, we obtain presentations for $GL(2, \mathbb{Z}_{2^k})$ and $GL(2, \mathbb{Z}_{p^k})$ $\forall$ $k \geq 1$.

For any finite field $\mathbb{F}_q$, we obtain generators of $GL(2, \mathbb{F}_q)$, and presentations for $GL(2, \mathbb{F}_{2^n})$, $GL(2, \mathbb{F}_{p^n})$ $\forall$ $n \geq 1$, in terms of Lie regular units.

We show that Lie regular elements also exist in $\mathcal{M}(4, R)$, where $\mathcal{M}(4, R)$ is the ring of 4×4 matrices over a commutative ring R with unity. We also show that these elements generate $GL(4, \mathbb{Z}_n)$ $\forall$ $n > 1$. We extend the result of existence of Lie regular elements to $\mathcal{M}(2n, R)$ for $n > 2$.

In this thesis, we also study about units in some group algebras. For this discussion, we compute the Wedderburn decomposition of $\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)}$ for a group G, and a finite field $\mathbb{F}_q$. More precisely, we discuss units of the group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_3)$, where $\mathbb{F}_q$ denotes a finite field of characteristic p, having q elements. We also discuss units of group algebra $\mathbb{F}_q G$ for all non-abelian groups of order 24 over a finite field $\mathbb{F}_q$ of characteristic 2 with $q = 2^k$ elements.

सारांश

यह शोध प्रबंध रिंग और ग्रुप रिंग के यूनिट ग्रुप पर है|. हमने जनरल लीनियर ग्रुप के जनरेटरों और प्रेज़ेंटेशनों पर धयान केन्द्रित किया है|. प्रत्येक $n > 1$ के लिए हमने $GL(2, \mathbb{Z}_n)$ के ली रेगुलर जनरेटरों का एक सेट निकाला है|. यह शर्मा आदि द्वारा किये गए कार्य का सामान्यकरण है|. तदउपरांत, प्रत्येक $k \geq 1$ के लिए हमने $GL(2, \mathbb{Z}_{2^k})$ और $GL(2, \mathbb{Z}_{p^k})$ की प्रेजेंटेशन दीं हैं|.

प्रत्येक फाइनाइट फील्ड $\mathbb{F}_q$, के लिए हमने $GL(2, \mathbb{F}_q)$ के ली रेगुलर यूनिट्स के रूप में जेनरेटर्स और प्रत्येक $n \geq 1$ के लिए $GL(2, \mathbb{F}_{2^n})$, $GL(2, \mathbb{F}_{p^n})$ की ली रेगुलर यूनिट्स के रूप में प्रेजेंटेशन निकाली हैं|.

हमने दर्शाया है, कि ली रेगुलर एलिमेंट्स $M(4, R)$ में भी उपस्थित हैं|. यहाँ $M(4, R)$, 4×4 आव्यूहों का रिंग है और R यूनिटी रखने वाला एक कम्युटेटिव रिंग है|. हमने यह भी दर्शाया है कि, प्रत्येक $n > 1$ के लिए ये एलिमेंट्स $GL(4, \mathbb{Z}_n)$ को जनरेट करते हैं|. अपने निष्कर्ष को आगे बढ़ाते हुए हमने दिखाया है कि, प्रत्येक $n > 2$ के लिए ली रेगुलर एलिमेंट्स $M(2n, R)$ में भी उपस्थित हैं|.

इस शोध प्रबंध में हमने कुछ ग्रुप अल्जेब्राओं के यूनिट एलिमेंट्स का भी अध्ययन किया है|. इस विचार-विमर्श के लिए हमने $\mathbb{F}_qG/J(\mathbb{F}_qG)$ कि वेडेरबर्न डिकम्पोसिशन निकाली है, जहाँ G एक ग्रुप और $\mathbb{F}_q$ एक फाइनाइट फील्ड है|. विशेष रूप से, हमने ग्रुप अल्जेब्रा $\mathbb{F}_qSL(2, \mathbb{Z}_3)$ के यूनिट एलिमेंट्स का अध्ययन किया है, जहाँ $\mathbb{F}_q$ कैरेक्टरिस्टिक p का एक फाइनाइट फील्ड है जिसमें q एलिमेंट्स हैं|. हमने कार्डीनेलिटी 24 के प्रत्येक नॉन-अबेलियन ग्रुप के लिए ग्रुप अल्जेब्रा $\mathbb{F}_qG$ के यूनिट एलिमेंट्स का भी अध्ययन किया है, जहाँ $\mathbb{F}_q$ कैरेक्टरिस्टिक 2 का $q = 2^k$ एलिमेंट्स वाला एक फाइनाइट फील्ड है|.

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List of Symbols

In the following notations, X is a set, G is a group, R is a ring and $\mathbb{F}$ is a field. Also p, q, s, l, n denote positive integers.

Symbol	Meaning
$\forall$	for all
$\in$	belongs to
$\subseteq$	subset or equal
$\subsetneq$	proper subset
$\cup$, $\cap$	union, intersection
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the ring of integrs modulo n

Symbol	**Meaning**
ϕ	Euler's totient functon
$\mathcal{M}(n, R)$	the ring of all $n \times n$ matrices over R
$GL(n, R)$	General linear group of degree n over R
$SL(n, R)$	Special linear group of degree n over R
I_n	$n \times n$ identity matrix
$char\ \mathbb{F}$	characteristic of the field $\mathbb{F}$
$\mathbb{F}_q$	finite field of characteristic p with q elements
$\mathbb{F}_q^*$	$\mathbb{F}_q \backslash \{0\}$
$\square$	end of a proof
$p \nmid n$	p does not divide n
$\lvert X \rvert$	cardinality of the set X
$o(G)$	order of the group G
$H \leq G$	H is a subgroup of G
$G \times H$	direct product of G and H
$\cong$	is isomorphic to
$\equiv$	is congruent to
$R \oplus S$	direct sum of rings R and S
$ker\ f$	kernel of the homomorphism f
$\times$	direct product
$\prod$	product of integers and also product of matrices
G/H	quotient of G by H
$[x]$	conjugacy class of $x \in G$
$\mathcal{U}(R)$	unit group of R
$\mathcal{Z}(R)$	center of R

Symbol	**Meaning**
$J(R)$	Jacobson radical of R
(q, s)	the greatest common divisor of q and s
$ord_s(q)$	multiplicative order of q modulo s, $(q, s) = 1$
$G \rtimes H$	semidirect product of G and H
C_n	cyclic group group of order n
Q_8	quaternion group of order 8
S_n	symmetric group on n symbols
A_n	alternating group on n symbols
D_{2n}	dihedral group of order $2n$
Dic_n	dicyclic group of order $4n$