IDENTITIES RELATED TO DERIVATIONS IN CERTAIN RINGS

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Identities Related to Derivations in Certain Rings"** submitted by **Mr. Vishal Kumar Yadav** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

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Abstract

Additive mappings play crucial role in ring theory. Derivation is also a particular type of additive mapping. A mapping d on a ring R is said to be a derivation if d is additive and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In ring theory, there are two important classes, the class of commutative rings, and the class of non commutative rings, in each of which we study various algebraic properties. A classical problem of ring theory is to find combinations of properties that force a ring to be commutative. Derivation on rings is also a tool to obtain conditions for commutativity of a ring, and thus helps in the study of structure of a ring. Though, the study of derivations was initiated long ago but it got impetus when Posner proved two pioneer results which establish relations between derivations, and commutativity of rings. Many authors' have extended Posner's results by generalizing derivations such as generalized derivation, Jordan derivation, (α, β)-derivation, symmetric n-derivation,..., etc and by taking algebraic conditions on particular subsets of a ring such as ideal, Lie ideal, Jordan ideal,..., etc. As we know that a non central Lie ideal contains all commutators of a non zero ideal of a ring of characteristic not 2. Motivated by this, Wong initiated study of derivation on multilinear polynomials. In this thesis, we study identities related to derivations acting on multilinear polynomials.

A mapping $D : R \times R \to R$ is said to be symmetric if $D(x_1, x_2) = D(x_{\pi(1)}, x_{\pi(2)})$ for all $x_1, x_2 \in R$, and for all permutation $\pi \in S_2$. The symmetric mapping D is said to be a bi-derivation if it is a derivation in each argument. From an additive commuting mapping, we can always find a bi-derivation. The notion of bi-derivation was given by Maksa. Motivated by bi-derivations, Park has introduced the concept of a symmetric n-derivation and generalized Posner's theorems on derivations to n-derivations. Continuing this study, we have obtained results for symmetric skew n-derivations on certain rings.

Let d be a derivation of R. Then an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, is said to be a generalized derivation on R associated with derivation d. This notion has been introduced by Brešar. Many authors, like Hvala, Lee, Sharma, Ashraf,..., etc. have studied algebraic identities related to generalized derivations and established relation between structure of a ring and generalized derivations. In this thesis, we have studied structure of a ring with the help of generalized derivations with appropriate algebraic conditions on suitable subsets of rings such as ideal, Lie ideal, Jordan ideal,..., etc.

Posner proved that the left annihilator of the image set of a non zero derivation in prime ring is zero. Several authors like Brešar, Filippis, Dhara, Sharma,..., etc. have studied left annihilator of identity related to derivations and generalized derivations on certain rings such as prime, semiprime,..., etc. In this sequence, we have examined the left annihilator of identity involving derivations and generalized derivations on prime rings and semiprime rings.

सार

योगात्मक फलन वलय सिद्धांतो मे महत्वपूर्ण भूमिका निभाते है | डेरिवेशन भी एक प्रकार का योगात्मक फलन है | एक फलन d एक वलय R का डेरिवेशन कहलाता है यदि d योगात्मक है तथा d(xy)=d(x)y+xd(y), जहाँ सभी x, y वलय R मे है | वलय सिद्धांतो मे दो महत्वपूर्ण वर्ग होते है एक विनिमय वलय और दूसरा गैर विनिमय वलय, जिसके प्रत्येक वर्ग मे हम विभिन्न बीजगणितीय विशेषताओ का अध्ययन करते है | वलय सिद्धांतो मे एक पारंपरिक प्रशन यह है कि उन गुण को खोजना जिसमे वलय विनिमय होता है | वलयो पर डेरिवेशन भी एक उपकरण है जिससे वलय के विनिमय होने की शर्तो को प्राप्त करते है तथा वलय की संरचना का अध्ययन करते है | हालाँकि डेरिवेशन का अध्ययन बहुत समय पहले शुरू किया गया था किंतु जब पासनर ने दो पथ प्रदर्शक सिद्धांतो को सिद्ध किया जो की डेरिवेशन और वलय के विनिमय होने के बीच संबंध स्थापित करते है, तब इसे अधिक प्रोत्साहन मिला | अनेक लेखको ने पासनर के सिद्धांतो का व्यापकीकरण, डेरिवेशन के व्यापकीकरण, जैसे व्यापक डेरिवेशन, जॉर्डन डेरिवेशन, (आल्फा,बीटा)-डेरिवेशन, सममित n-डेरिवेशन, ...,आदि तथा बीजगणितीय शर्तो को वलय के उपसमुच्चय जैसे आइडीयल, ली आइडीयल, जॉर्डन आइडीयल, ...,आदि के मार्फत से किया | जैसा की हम जानते है कि एक गैर केंद्रीय ली आइडीयल, एक वलय जिसकी char. 2 नही है, के सभी गैर शून्य आइडीयल के विनिमयो को सम्मलित करता है | इससे प्रेरित होकर वाँग ने डेरिवेशन का बहुरेखिक बहुपदो पर अध्ययन प्रारंभ किया | इस थीसिस मे हम उन सर्वसामिकाओ का अध्ययन करेंगे जिनमे डेरिवेशन बहुरेखिक बहुपदो पर है |

एक फलन $D: R \times R \rightarrow R$ सममित कहलाता है, यदि $D(x_1, x_2) = D(x_{\pi(1)}, x_{\pi(2)})$, जहाँ सभी x_1,x_2 वलय R, तथा π क्रमचय समूह S_2 मे है | सममित फलन D वाई -डेरिवेशन कहलाता है यदि यह प्रत्येक चर मे डेरिवेशन है | हम प्रत्येक योगात्मक फलन से वाई -डेरिवेशन प्राप्त कर सकते है | वाई -डेरिवेशन की अवधारणा को मास्का दुवारा दिया गया| वाई -डेरिवेशन से प्रेरित होकर पार्क ने n-डेरिवेशन की अवधारणा को प्रस्तुत किया तथा पासनर के सिद्धांतो का व्यापकीकरण किया| इस अध्ययन को जारी रखते हुए हमने इस्केव सममित डेरिवेशन के सिद्धांतो को कुछ वलयो पर सिद्ध किया है |

माना एक वलय R पर d एक डेरिवेशन है | तब एक योगात्मक फलन F: R → R को डेरिवेशन d से संवद्धित व्यापक डेरिवेशन कहा जाएगा यदि F(xy)= F(x)y+xd(y), जहाँ सभी x, y वलय R मे है | इस अवधारणा को ब्रेसर द्वारा दिया गया | अनेको लेखको जैसे हवला, ली, शर्मा,...,आदि ने व्यापक डेरिवेशन से संवद्धित सर्वसामिकाओ का अध्ययन किया तथा व्यापक डेरिवेशन और वलय की संरचना के बीच संबंध स्थापित किया | इस थीसिस मे हम ने उचित बीजगणितीय शर्तो के साथ व्यापक डेरिवेशन का वलय के उपसमुच्चय जैसे आइडीयल, ली-आइडीयल, जॉर्डन आइडीयल,...,आदि पर अध्ययन किया है |

पासनर ने सिद्ध किया था की एक प्राइम वलय पर एक गैर शून्य डेरिवेशन के पारस का अनिहिलेटर शून्य होता है | अनेको लेखको जैसे ब्रेसर, शर्मा, धारा,..,आदि ने कुछ वलयो जैसे प्राइम, सेमी प्राइम,..,आदि पर डेरिवेशन और व्यापक डेरिवेशन से संवद्धित सर्वसामिकाओ के वायें अनिहिलेटर का अध्ययन किया | इसी क्रम को जारी रखते हुए हम ने प्राइम तथा सेमी प्राइम वलय पर डेरिवेशन और व्यापक डेरिवेशन से संवद्धित सर्वसामिकाओ के वायें अनिहिलेटर का अध्ययन किया है |

Contents

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{F}$	a field
$x \in X$	x is a member of X
$x \notin X$	x is not a member of X
$A \subset R$	A is a proper subset of R
$A \subseteq R$	A is a subset of R
$A \nsubseteq R$	A is not a subset of R
$\mathbb{Z}_n$	the group of integers modulo n
$=$	equal to
$\neq$	not equal to
S_n	permutation group on n symbols
s_n	standard polynomial on n symbols
$Z(R)$	center of a ring R
$char(R)$	characteristic of R
d	derivation on R
F	generalized derivation on R

U	Utumi quotient ring
Q	Martindale quotient ring
C	extended centroid of a prime ring R
$S = RC$	centre closure of a prime ring R
$[x, y]$	Lie product $xy - yx$ of x and y
$x \circ y$	Jordan product $xy + yx$ of x and y
L	a Lie ideal of a ring R
J	a Jordan ideal of a ring R
$ann(R)$	annihilator of R
$\cup, \cap$	union, intersection
$A \times B$	cartesian product of A and B
$A \oplus B$	A direct sum B
$A \cong B$	A is isomorphic to B
PI	polynomial identity
GPI	generalized polynomial identity
$Soc(R)$	socle of R