NUMERICAL SOLUTION OF SOME SINGULARLY PERTURBED PROBLEMS HAVING BOUNDARY AND INTERIOR LAYERS

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AUGUST 2017

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by

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Submitted

in fulfilment of the requirements of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Delhi
August 2017

Dedicated to My Parents

Certificate

This is to certify that the thesis entitled **Numerical Solution of Some Singularly Perturbed Problems Having Boundary and Interior layers** submitted by **Ms Sheetal Chawla** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
August 2017

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Acknowledgements

Innumerable appreciation and deepest gratitude for the support and help of the following persons who in one way or another have contributed in making this study possible.

I offer my sincerest gratitude to my thesis supervisor, Prof. S. Chandra Sekhara Rao for his patience, wisdom, endless support, valuable suggestions, inspiration and above all for his constant encouragement for carrying out this work. Dr Rao introduced me to a very interesting area of mathematics. He always gave me freedom to explore on new ideas. He taught me how to question thoughts and express ideas. His constant support throughout the program helped me overcome many situations and finish this thesis. The knowledge and wisdom I have gained from him will guide me forever in education and in life.

I would like to give special thanks to Prof R.K. Sharma as a Chairperson SRC (Student Research Committee) and my SRC members Dr K. Sreenadh and Prof Madhusudan Rao. I am grateful to all faculty members of Department of Mathematics, IIT Delhi, for their cooperation and support.

I am thankful to IIT Delhi authorities for providing me the necessary facilities and teaching assistantship for smooth completion of my research work and also for providing me the financial support to attend the NAA2012 conference in Bulgaria. This gave me an opportunity to interact with extremely knowledgeable professionals and also increased my confidence. I would like to thank the Council for Scientific and Industrial Research(CSIR) for providing me financial assistance by awarding me research fellowship during my Ph.D.

I also like to express a heartfelt thank to all my friends Poonam, Manju, Manisha, Swati sidana, Rachana, Rabia, Parvesh, Sudhakar for all the support and fun when I needed a reprieve from the intensity of research. Many thanks to my seniors Dr Mukesh, Dr Sunil and Varsha Srivastava for their support, suggestions and cooperation.

Above all I am forever indebted to my entire family for their love, support and encouragement through this journey. I would like to thank my parents, Smt. Nirmala Rani and Sh. Ram lal for their life long support, love and sacrifices. It is also blessings of my mother in law Smt. Satwanti which motivated me towards the successful completion of this study. I am grateful to my brother Rajkumar and sisters Sangeeta and Neeru for their never ending motivation and belief in me. I would also like to thank my sister in laws Monika and Meenakshi and brother in laws Rajesh, Sunil and Jasbir for their Good wishes. I extend my love to Isha, Harshita, Vrinda, Anisha, Meera, Chirag, Raman and my son Rihaan(in eight months he cooperated so much, I have no words to explain) for their unconditional love.

I would also like to express all my thanks to a special person in my life; my husband Dr Jagbir Singh for always being there for me. Without his encouragement, understanding and support, it would not have been possible to complete this work.

Finally, I thank my Babaji for the passion, strength, perseverance, and the resources to complete this study.

New Delhi *Sheetal Chawla*

Abstract

This thesis is devoted to design and analyze Parameter-Uniform Numerical Methods for System of Some Singularly Perturbed Problems having Boundary and Interior Layers. These problems appear in many physical phenomenon, such as semiconductor device modeling, investigation of diffusion process complicated by chemical reactions in electro analytic chemistry and in electrical networks.

To approximate the solution of system of $m(\geq 2)$ singularly perturbed reaction-diffusion equations with discontinuous source term a numerical method is constructed. The numerical method gives parameter-uniform numerical approximations of almost second order. With the suitable combinations of mesh functions and improving the bounds on the derivatives of the components of the solution we are able to achieve almost second order convergence theoretically also. We then extend the method to the systems of $m(\geq 2)$ singularly perturbed semilinear reaction-diffusion problems with discontinuous source term.

We next propose a numerical method for system of $m(\geq 2)$ singularly perturbed two dimensional reaction-diffusion problems with discontinuous source term in both x and y directions with equal diffusion parameters. We prove that the method is almost second order uniformly convergent, although along the point of discontinuities and at some transition points it is almost first order uniformly convergent. We construct suitable mesh functions to raise the order to two.

For the solution of system of two singularly perturbed convection-diffusion equations with discontinuous convection coefficient a numerical method is constructed. Due to discontinuity and the sign convection of the convection coefficient the boundary layers are appearing at both the ends. The numerical approximations obtained from this method are proved to be almost first order uniformly convergent.

We considered the systems of $m(\geq 2)$ singularly perturbed linear time dependent reaction-diffusion problems with discontinuous source term and different diffusion parameters. The numerical method gives parameter-uniform approximations of first order in time and almost second order in space.

A numerical method is constructed for a system of two singularly perturbed parabolic convection-diffusion problems with discontinuous convection coefficient. The numerical method developed is proved to be almost first order uniformly convergent both in time and space direction.

Numerical experiments are conducted on some test problems for all the developed numerical methods to validate the theoretical results.

<u>सारांश</u>

यह शोध प्रबन्ध सीमा और आंतरिक परतों वाले सिस्टम ऑफ सिंगुलर्ली पर्टर्ब्ड प्राब्लम्ज़ के लिए पैरामीटर-यूनिफ़ॉर्म न्यूमेरिकल विधियों के डिजाइन और विश्लेषण के लिए समर्पित है। ये समस्याएं कई भौतिक घटनाओं में दिखाई देती हैं, जैसे कि अर्धचालक डिवाइस मॉडलिंग, इलेक्ट्रो विश्लेषणात्मक रसायन विज्ञान में रासायनिक प्रतिक्रियाओं और विद्युत नेटवर्क में जटिल प्रसार प्रक्रिया की जांच।

सिस्टम ऑफ m (≥2) सिंगुलर्ली पर्टर्ब्ड रीऐक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म के सलूशन को अप्राक्समैट करने के लिए एक संख्यात्मक विधि का निर्माण किया गया है। संख्यात्मक विधि लगभग सेकन्ड ऑर्डर पैरामीटर-यूनिफ़ॉर्म अप्राक्समेशनस देती हैं। मेष फ़ंक्शनस के उपयुक्त संयोजनों और सलूशन के कम्पोनन्ट्स के डेरिवेटिवस पर बाउन्ड्ज़ को सुधारने के साथ हम सैद्धांतिक रूप से लगभग सेकन्ड ऑर्डर को प्राप्त करने में सक्षम हैं। फिर इस पद्धति का विस्तार सिस्टम ऑफ m (≥2) सिंगुलर्ली पर्टर्ब्ड सेमिलिनीअर रीऐक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म के लिए किया गया है।

आगे सिस्टम ऑफ m (≥2) सिंगुलर्ली पर्टर्ब्ड टू डिमेन्शनल रीऐक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म एक्स-वाई दोनों दिशाओं में और समान डिफ्यूशन परैमिटरस के लिए एक संख्यात्मक विधि प्रस्तावित की गई हैं। विधि लगभग सेकन्ड ऑर्डर यूनिफ़ॉर्म कन्वर्जन्ट हैं, हालांकि डिस्कान्टनूइटीस के बिंदु पर और कुछ ट्रैन्ज़िशन बिंदुओ पर लगभग फर्स्ट ऑर्डर यूनफॉर्म्ली कन्वर्जन्ट सिद्ध किया गया हैं। टू ऑर्डर बढ़ाने के लिए उपयुक्त मेष फ़ंक्शन का निर्माण किया गया हैं।

सिस्टम ऑफ टू सिंगुलर्ली पर्टर्ब्ड कन्वेक्शन डिफ्यूशन इक्वेशनस विद डिस्कन्टिन्यूअस कन्वेक्शन कोअफिशन्ट के सलूशन के लिए एक संख्यात्मक विधि का निर्माण किया गया है। डिस्कान्टिनूइटी और कन्वेक्शन कोअफिशन्ट के साइन कन्वेन्शन के कारण सीमाओं के दोनों किनारों पर परतें दिखाई दे रही हैं। इस विधि से प्राप्त न्यूमेरिकल अप्राक्समेशनस लगभग फर्स्ट ऑर्डर यूनफॉर्म्ली कन्वर्जन्ट हैं।

सिस्टम ऑफ m (≥2) सिंगुलर्ली पर्टर्ब्ड लिनीअर टाइम डिपेन्डन्ट रीऐक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म और डिफ्रन्ट डिफ्यूशन परैमिटरस कन्सिडर की गई हैं। संख्यात्मक विधि टाइम में फर्स्ट ऑर्डर और स्पैस में लगभग सेकन्ड ऑर्डर पैरामीटर-यूनिफ़ॉर्म अप्राक्समेशनस देती हैं।

सिस्टम ऑफ टू सिंगुलर्ली पर्टर्ब्ड पैराबालिक कन्वेक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस कन्वेक्शन कोअफिशन्ट के लिए एक संख्यात्मक विधि का निर्माण किया गया है। विकसित

संख्यात्मक विधि टाइम और स्पैस दोनों डाइरेक्शिन में लगभग फर्स्ट ऑर्डर यूनफॉम्र्ली कन्वर्जन्ट सिद्ध किया गया हैं।

सैद्धांतिक परिणामों को सत्यापित करने हेतु सभी विकसित न्यूमेरिकल तरीकों के लिए कुछ परीक्षण समस्याओं पर न्यूमेरिकल प्रयोग किए गयें हैं।

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ε	singular perturbation parameter
h	step size
Ω	given space variable domain
$\sigma_{\varepsilon_{l_k}}$	subdomain parameter on left
$\sigma_{\varepsilon_{r_k}}$	subdomain parameter on right
Ω_1	subdomain of Ω
Ω_2	subdomain of Ω
$G = \Omega \times (0, T]$	given domain for non-stationary problems
$G_1 = \Omega_1 \times (0, T]$	subdomain for G for non-stationary problems
$G_2 = \Omega_2 \times (0, T]$	subdomain for G for non-stationary problems
$C^s(\Omega), C^s(\Omega_1 \cup \Omega_2)$	function spaces
v_i	$v(x_i)$
$v_{p;i}$	$v_p(x_i)$
$\boldsymbol{v}_i$	$\boldsymbol{v}(x_i) = (v_{1;i}, \ldots, v_{m;i})$
$\boldsymbol{v}_{p;i}$	$\boldsymbol{v}_p(x_i) = (v_{p,1;i}, \ldots, v_{p,m;i})$
$\|v\|_{\overline{\Omega}^N}$	$\max_{x_i \in \overline{\Omega}^N} \|v(x_i)\|$

$\|\boldsymbol{v}\|_{\overline{\Omega}^N}$	$\max\{\|v_1\|_{\overline{\Omega}^N}, \ldots, \|v_m\|_{\overline{\Omega}^N}\}$
$C, C_s, s = 0, 1, ...$	generic positive constants, independent of ε, N
$\boldsymbol{C}, \boldsymbol{C}_s, s = 0, 1, ...$	generic positive constant vectors