

**NUMERICAL SOLUTION OF SOME
SINGULARLY PERTURBED PROBLEMS
HAVING BOUNDARY AND INTERIOR LAYERS**

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**DEPARTMENT OF MATHEMATICS
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SINGULARLY PERTURBED PROBLEMS
HAVING BOUNDARY AND INTERIOR LAYERS**

by

SHEETAL CHAWLA

Department of Mathematics

Submitted

in fulfilment of the requirements of the degree of Doctor of Philosophy

to the



**Indian Institute of Technology Delhi
August 2017**

Dedicated to My Parents

Certificate

This is to certify that the thesis entitled **Numerical Solution of Some Singularly Perturbed Problems Having Boundary and Interior layers** submitted by Ms **Sheetal Chawla** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
August 2017

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Sheetal Chawla

Abstract

This thesis is devoted to design and analyze Parameter-Uniform Numerical Methods for System of Some Singularly Perturbed Problems having Boundary and Interior Layers. These problems appear in many physical phenomenon, such as semiconductor device modeling, investigation of diffusion process complicated by chemical reactions in electro analytic chemistry and in electrical networks.

To approximate the solution of system of $m(\geq 2)$ singularly perturbed reaction-diffusion equations with discontinuous source term a numerical method is constructed. The numerical method gives parameter-uniform numerical approximations of almost second order. With the suitable combinations of mesh functions and improving the bounds on the derivatives of the components of the solution we are able to achieve almost second order convergence theoretically also. We then extend the method to the systems of $m(\geq 2)$ singularly perturbed semilinear reaction-diffusion problems with discontinuous source term.

We next propose a numerical method for system of $m(\geq 2)$ singularly perturbed two dimensional reaction-diffusion problems with discontinuous source term in both x and y directions with equal diffusion parameters. We prove that the method is almost second order uniformly convergent, although along the point of discontinuities and at some transition points it is almost first order uniformly convergent. We

construct suitable mesh functions to raise the order to two.

For the solution of system of two singularly perturbed convection-diffusion equations with discontinuous convection coefficient a numerical method is constructed. Due to discontinuity and the sign convection of the convection coefficient the boundary layers are appearing at both the ends. The numerical approximations obtained from this method are proved to be almost first order uniformly convergent.

We considered the systems of $m(\geq 2)$ singularly perturbed linear time dependent reaction-diffusion problems with discontinuous source term and different diffusion parameters. The numerical method gives parameter-uniform approximations of first order in time and almost second order in space.

A numerical method is constructed for a system of two singularly perturbed parabolic convection-diffusion problems with discontinuous convection coefficient. The numerical method developed is proved to be almost first order uniformly convergent both in time and space direction.

Numerical experiments are conducted on some test problems for all the developed numerical methods to validate the theoretical results.

सारांश

यह शोध प्रबन्ध सीमा और आंतरिक परतों वाले सिस्टम ऑफ सिंगुलर्ली पर्टर्ब्ड प्राब्लम्ज़ के लिए पैरामीटर-यूनिफ़ॉर्म न्यूमेरिकल विधियों के डिजाइन और विश्लेषण के लिए समर्पित है। ये समस्याएं कई भौतिक घटनाओं में दिखाई देती हैं, जैसे कि अर्धचालक डिवाइस मॉडलिंग, इलेक्ट्रो विश्लेषणात्मक रसायन विज्ञान में रासायनिक प्रतिक्रियाओं और विद्युत नेटवर्क में जटिल प्रसार प्रक्रिया की जांच।

सिस्टम ऑफ $m (\geq 2)$ सिंगुलर्ली पर्टर्ब्ड रीएक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म के सलूशन को अप्राक्समैट करने के लिए एक संख्यात्मक विधि का निर्माण किया गया है। संख्यात्मक विधि लगभग सेकन्ड ऑर्डर पैरामीटर-यूनिफ़ॉर्म अप्राक्समेशनस देती हैं। मेष फ़ंक्शनस के उपयुक्त संयोजनों और सलूशन के कम्पोनन्ट्स के डेरिवेटिवस पर बाउन्डज़ को सुधारने के साथ हम सैद्धांतिक रूप से लगभग सेकन्ड ऑर्डर को प्राप्त करने में सक्षम हैं। फिर इस पद्धति का विस्तार सिस्टम ऑफ $m (\geq 2)$ सिंगुलर्ली पर्टर्ब्ड सेमिलिनीअर रीएक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म के लिए किया गया है।

आगे सिस्टम ऑफ $m (\geq 2)$ सिंगुलर्ली पर्टर्ब्ड टू डिमेन्शनल रीएक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म एक्स-वाई दोनों दिशाओं में और समान डिफ्यूशन परैमिटरस के लिए एक संख्यात्मक विधि प्रस्तावित की गई है। विधि लगभग सेकन्ड ऑर्डर यूनिफ़ॉर्म कन्वर्जन्ट हैं, हालांकि डिस्कान्टिन्यूइटीस के बिंदु पर और कुछ ट्रैन्ज़िशन बिंदुओ पर लगभग फर्स्ट ऑर्डर यूनिफ़ॉर्मली कन्वर्जन्ट सिद्ध किया गया है। टू ऑर्डर बढ़ाने के लिए उपयुक्त मेष फ़ंक्शन का निर्माण किया गया है।

सिस्टम ऑफ टू सिंगुलर्ली पर्टर्ब्ड कन्वेक्शन डिफ्यूशन इक्वेशनस विद डिस्कन्टिन्यूअस कन्वेक्शन कोअफिशन्ट के सलूशन के लिए एक संख्यात्मक विधि का निर्माण किया गया है। डिस्कान्टिन्यूइटी और कन्वेक्शन कोअफिशन्ट के साइन कन्वेन्शन के कारण सीमाओं के दोनों किनारों पर परतें दिखाई दे रही हैं। इस विधि से प्राप्त न्यूमेरिकल अप्राक्समेशनस लगभग फर्स्ट ऑर्डर यूनिफ़ॉर्मली कन्वर्जन्ट हैं।

सिस्टम ऑफ $m (\geq 2)$ सिंगुलर्ली पर्टर्ब्ड लिनीअर टाइम डिपेन्डन्ट रीएक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस सोर्स टर्म और डिफ्रन्ट डिफ्यूशन परैमिटरस कन्सिडर की गई हैं। संख्यात्मक विधि टाइम में फर्स्ट ऑर्डर और स्पेस में लगभग सेकन्ड ऑर्डर पैरामीटर-यूनिफ़ॉर्म अप्राक्समेशनस देती हैं।

सिस्टम ऑफ टू सिंगुलर्ली पर्टर्ब्ड पैराबालिक कन्वेक्शन डिफ्यूशन प्राब्लम्ज़ विद डिस्कन्टिन्यूअस कन्वेक्शन कोअफिशन्ट के लिए एक संख्यात्मक विधि का निर्माण किया गया है। विकसित

संख्यात्मक विधि टाइम और स्पैस दोनों डाइरेक्शन में लगभग फर्स्ट ऑर्डर यूनफॉर्मली कन्वर्जेंट सिद्ध किया गया है।

सैद्धांतिक परिणामों को सत्यापित करने हेतु सभी विकसित न्यूमेरिकल तरीकों के लिए कुछ परीक्षण समस्याओं पर न्यूमेरिकल प्रयोग किए गये हैं।

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List of Symbols

ε	singular perturbation parameter
h	step size
Ω	given space variable domain
$\sigma_{\varepsilon l_k}$	subdomain parameter on left
$\sigma_{\varepsilon r_k}$	subdomain parameter on right
Ω_1	subdomain of Ω
Ω_2	subdomain of Ω
$G = \Omega \times (0, T]$	given domain for non-stationary problems
$G_1 = \Omega_1 \times (0, T]$	subdomain for G for non-stationary problems
$G_2 = \Omega_2 \times (0, T]$	subdomain for G for non-stationary problems
$C^s(\Omega), C^s(\Omega_1 \cup \Omega_2)$	function spaces
v_i	$v(x_i)$
$v_{p;i}$	$v_p(x_i)$
\mathbf{v}_i	$\mathbf{v}(x_i) = (v_{1;i}, \dots, v_{m;i})$
$\mathbf{v}_{p;i}$	$\mathbf{v}_p(x_i) = (v_{p,1;i}, \dots, v_{p,m;i})$
$\ v\ _{\overline{\Omega}^N}$	$\max_{x_i \in \overline{\Omega}^N} v(x_i) $

$\ \mathbf{v}\ _{\overline{\Omega}^N}$	$\max\{\ v_1\ _{\overline{\Omega}^N}, \dots, \ v_m\ _{\overline{\Omega}^N}\}$
$C, C_s, s = 0, 1, \dots$	generic positive constants, independent of ε, N
$\mathbf{C}, \mathbf{C}_s, s = 0, 1, \dots$	generic positive constant vectors