On certain exact solutions of a generalized K-dV-Burger type equation via Isovector method – I

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Abstract

A generalized K-dV-Burger type equation

\[ u_t + a(u^p)_x - \alpha (u^q)_x + \beta (u^r)_{xxx} - g(x)u^q = 0, \]

where \( m, n, p, q, a, \alpha, \beta \) are constants and \( g(x) \) is an arbitrary function, has been analyzed through isovectors method. After constructing the components of the isovector field, the invariant groups of transformations are determined by means of the orbital equations. These are further utilized to derive certain exact solutions corresponding to different cases in terms of the parameters of the equation.

1. Introduction

The study of nonlinear diffusion and nonlinear convection-diffusion problems which find numerous applications in areas of science and engineering, has been the subject of investigations for decades by mathematicians and physicists alike [1–7]. There have also been important contributions of varying nature on the phenomenon of nonlinear diffusion with absorption which occurs in spatial diffusion of biological populations and in a number of chemical diffusion processes where the physical structure of the medium changes with concentration [8–10]. A number of exact
solutions to these equations and some of their variants can be found in literature (see for example: [1,2,5,6,9,11]). Herein we consider a quasilinear equation in the following generalized form:

\[ u_t + a(u^a)_x - \alpha(u^p)_{xx} + \beta(u^q)_{xxx} - g(x)u^q = 0, \quad (1.1) \]

where \( m, n, p, q, a, \alpha, \beta \) are constants and \( g(x) \) is an arbitrary function.

It may be noted that Eq. (1.1) incorporates the effect of dispersion besides those mentioned above. For \( a = 0 = \beta, \alpha = 1 \), Eq. (1.1) reduces to

\[ u_t = (u^p)_{xx} + g(x)u^q, \quad n \neq 0 \quad (1.2) \]

which serves as a simple mathematical model for various physical problems such as to describe the flow of liquids in porous media or the transport of thermal energy in plasma. Eq. (1.2) was examined by Gandarias [12] for symmetry reductions and exact solutions using the nonclassical method developed by Bluman and Cole [13].

Further, for \( x = 0, \beta = 1 \) and \( g(x) \equiv 0 \) Eq. (1.1) assumes the following form:

\[ u_t + a(u^a)_x + (u^p)_{xxx} = 0. \quad (1.3) \]

Eq. (1.3) exhibits a number of remarkable dispersive effects and has been analyzed by Rosenau [14] for nonanalytic solitary waves for some special cases in terms of the parameters \( m \) and \( p \).

One can now regard Eq. (1.1) as a natural generalization of Eqs. (1.2) and (1.3). In this work we have carried over the isovector method which is an alternative and systematic approach to obtain the generators of the symmetries of a single or system of partial differential equations. The method was first introduced by Harrison and Estabrook [15] in their fundamental paper and later developed by Edelen [16]. For some interesting applications of isovector method we suggest our reader to browse through the references [11,17–20]. To provide solutions to Eq. (1.1) for the case \( \beta = 0 \), we have analyzed it using the conventional symmetry approach based on the application of Fréchet derivative to determine the symmetries and some special exact solutions. The study has also yielded some new results besides recovering some of the available results for various simpler versions of Eq. (1.1) as obtained by assigning some particular values to the various parameters occurring in the equation. These results constitute the subject for our next communication [21].

The present study has been planned as follows. In Section 2, we deal with the determination of isovectors for the equation under consideration. The similarity solutions for various particular cases by means of the isovectors listed in the form of tables in Section 2 are presented in Section 3. Finally, we record the concluding remarks in Section 4.

2. Isovector method

2.1. Determination of isovector field

In order to apply the method we introduce the variables \( T, y \) and \( z \) as follows

\[ T = u_t, \quad y = u_x, \quad z = u_{xx}. \quad (2.1) \]
Eq. (1.1) can equivalently be described by the following set of differential forms, provided $\beta \neq 0$,
\begin{equation}
\alpha_1 = du - T \, dt - y \, dx,
\end{equation}
\begin{equation}
\alpha_2 = dy \wedge \, dt - z \, dx \wedge \, dt,
\end{equation}
\begin{equation}
\alpha_3 = -\beta pu^{p-1} \, dz \wedge \, dt + \psi \, dx \wedge \, dt,
\end{equation}
where
\[
\psi = g(x)u^a - T - am \, u^{n-1}y + \alpha\{u(n-1)u^{n-3}y^2 + mu^{n-1}z\}
\]
\[
- \beta\{p(p-1)(p-2)u^{p-3}y^3 + 3p(p-1)u^{p-2}yz\}
\]
and $\alpha_1, \alpha_2$ and $\alpha_3$ are 1, 2, 2-forms, respectively and $\wedge$ denotes the exterior product of differential forms.

Let $I = \{\alpha_1, \alpha_2, \alpha_3, \, dx_1\}$. It can be seen that $\alpha_1 = \alpha_2 = \alpha_3 = dx_1 = 0$ lead to Eqs. (1.1) and (2.1), whereas $dx_1 = 0$ results in the integrability condition $u_{\alpha} = u_{\alpha}$. Hence, it is proved that $I$ is an ideal which is closed, i.e., $dI \subset I$. For, we have
\[
dl \alpha_2 = \left[-\frac{1}{\beta p}u^{1-p} \, dx\right] \wedge \alpha_3
\]
and
\[
dl \alpha_3 = (f_1 \, dz \wedge \, dt + f_2 \, dx \wedge \, dt) \wedge \alpha_1 - f_3 \, dx \wedge \alpha_2 + (f_4 y - f_4) \, dx_1
\]
where
\[
f_1 = \beta p(p-1)u^{p-2},
\]
\[
f_2 = ag(x)u^{1-1} - am(m-1)y u^{n-2} + x\{u(n-1)(n-2)y^2u^{p-3} - \beta p(p-1)(p-2)y^3u^{p-4}
\]
\[
+ \alpha\{u(n-1)u^{n-3}z - 3\beta p(p-1)(p-2)yzu^{p-3}\}
\]
\[
f_3 = -amu^{p-1} + 2x\{u(n-1)y u^{p-2} - 3\beta p(p-1)(p-2)y^2u^{p-3} - 3\beta p(p-1)zu^{p-2}\}
\]
and
\[
f_4 = amu^{p-1} - 3\beta p(p-1)u^{p-2}.
\]

Next, we define a vector field $V$ over a six dimensional space $E_6 \equiv E_6(t,x,u,y,z,T)$ of the form
\begin{equation}
V \equiv V' \partial_t + V^x \partial_x + V^n \partial_u + V^y \partial_y + V^z \partial_z + V^T \partial_T,
\end{equation}
where $V'$, $V^x$, $V^n$, etc. are the components of $V$ in the direction as denoted by their respective superscripts and these are all smooth functions of the variables $t,x,u,y,z,T$. To find out the
infinitesimal generators of symmetry groups connected with Eq. (1.1) we need to determine the isovector field of $I$. Alternatively, we determine such vector fields that $I$ is stable under transport along their orbits, i.e.

$$\mathcal{L}_V I(x_1, x_2, x_3, dx_1) \subset I(x_1, x_2, x_3, dx_1)$$

where $\mathcal{L}_V$ denotes the Lie derivative w.r.t. the vector field $V$. The necessary and sufficient condition for $V$ to be an isovector field of $I$ is that the Lie derivative of every generator of $I$ should remain in $I$. For the case under consideration the transport property of exterior differential forms $\omega_i$, $i = 1, 2, 3$, can be expressed as

$$\mathcal{L}_V (\omega_1) = \lambda_1 \omega_1,$$

$$\mathcal{L}_V (\omega_2) = \xi_1 \wedge \omega_1 + \lambda_2 d\omega_1 + \lambda_3 \omega_2 + \lambda_4 \omega_3,$$

$$\mathcal{L}_V (\omega_3) = \xi_2 \wedge \omega_1 + \lambda_5 d\omega_1 + \lambda_6 \omega_2 + \lambda_7 \omega_3,$$

where $\xi_i$, $i = 1, 2$ are arbitrary 1-forms and $\lambda_j$, $j = 1, 2, \ldots, 7$ are arbitrary functions of $t, x, u, y, z, T$. Further, the Lie derivative of a differential form $\omega$ is given by

$$\mathcal{L}_V (\omega) = d(V \lrcorner \omega) + V \lrcorner d\omega,$$

where $d$ stands for exterior differentiation and $\lrcorner$ denotes the inner multiplication of forms. Thus, Eq. (2.7) now reads as

$$d(V \lrcorner \omega_1) + V \lrcorner d\omega_1 = \lambda_1 \omega_1.$$

Let $V \lrcorner \omega_1 = G(t, x, u, y, z, T)$. On substituting $\omega_1$ from Eq. (2.2) into Eq. (2.10) and performing expansion under exterior differentiation and inner multiplication operations and collecting the coefficients of similar 1-forms and equating them to zero and eliminating the arbitrary function $\lambda_1$ from the resulting equations, we get the following:

$$V^t = -G_t,$$

$$V^x = -G_x,$$

$$V^u = G - T G_t - y G_y,$$

$$V^y = G_x + y G_u,$$

$$V^T = G_t + T G_u,$$

$$G_z = 0.$$  \hfill (2.11)

Further, let

$$\xi_i = A_i \, dt + B_i \, dx + C_i \, du + D_i \, dy + E_i \, dz + F_i \, dT, \quad i = 1, 2.$$

From Eq. (2.8), on equating now the coefficients of similar 2-forms to zero and then eliminating the arbitrary functions $A_1, B_1, \ldots, A_2, \ldots$ etc. we arrive at the following new additional equations:
\[ G_T = 0, \quad (2.12a) \]
\[ G_{\alpha\tau} + yG_{\alpha\tau} + zG_{\alpha\tau} = 0, \quad (2.12b) \]
\[ V^z = G_{\alpha\alpha} + 2yG_{\alpha\alpha} + 2zG_{\alpha\alpha} + zG_{\alpha\alpha} + 2yG_{\alpha\alpha} + y^2G_{\alpha\alpha} + z^2G_{\alpha\alpha}. \quad (2.12c) \]

It is worth mentioning here at this point that during the process of simplifications of set of equations obtained from Eq. (2.8) we make use of Eq. (2.11), wherever required.

On making similar substitutions in Eq. (2.9) and performing expansions as stated earlier and collecting the coefficients of similar 2-forms and equating them to zero and then eliminating the functions \(A_2, B_2, \lambda_3, \ldots\) etc. and again making use of the equations so far obtained and as listed in Eqs. (2.11)–(2.12c) we arrive at the following additional equations:

\[ u[-G_T - TG_x - g' uG_y + f_2(G - TG_T - yG_T) + f_3(G_x + yG_u)] + u f_3 [G_{xx} + 2yG_{xy} + 2zG_{xy} + zG_{ux} + 2yG_{uy} + y^2G_{uu} + z^2G_{xy} - 3zG_{ux} + 3y^2G_{uy} + 3zG_{uy} + 3yG_{uy} + y^2G_{uu} + 3z^2G_{xy}] \]
\[ + 6yG_{uy} + 3y^2G_{uu} + 3z^2G_{xy} + y(3G_{xy} + 3zG_{xy} + 3yG_{xy} + y^2G_{uu} + 3z^2G_{xy}) \]
\[ + z(3G_{xy} + 3zG_{xy} + z^2G_{xy})] + \psi [(p - 1)(2T G_T G + yG_T) \]
\[ - u(3G_{xy} + 3yG_{xy} + 3zG_{xy} + G_T)] = 0, \quad (2.13a) \]
\[ u(G_{x\tau} + yG_{u\tau}) + (p - 1)yG_T = 0, \quad (2.13b) \]
\[ G_T = 0, \quad (2.13c) \]
\[ G_{\alpha\tau} + 2yG_{\alpha\tau} + zG_{\alpha\tau} + y^2G_{\alpha\tau} = 0, \quad (2.13d) \]

where \(G = G(t, x, u, y, T)\), and \(\psi\) represents the coefficient of \(dx \wedge dt\) in Eq. (2.4).

Our next task is to solve Eqs. (2.12a,b) and (2.13) to find out the form of \(G\) which in turn would furnish us the components of the vector field through Eqs. (2.11) and (2.12c).

To this end we first note that Eqs. (2.12a,b) and (2.13c) together yield \(G\) to be of the form given as under

\[ G = \phi_1(t) T + \phi_2(t, x, u, y), \quad (2.14) \]

where \(\phi_1\) and \(\phi_2\) are two functions which remain to be determined. Consequently, we observe that Eq. (2.13d) is identically satisfied and we are left with Eq. (2.13a), and Eq. (2.13b) reducing to

\[ (p - 1)G_T = 0. \]

This brings forth the following two cases to consider:

(i) \(p \neq 1\) and (ii) \(p = 1\).
In each case $G$ is determined with the help of Eq. (2.13a). While performing these calculations there arise numerous possibilities to deal with in terms of the parameters of Eq. (1.1). In order to avoid the details of these calculations, and to facilitate the reader, the final results are summarized in the form of Tables 1 and 2 corresponding, respectively, to the two cases described above.

3. Similarity reductions and exact solutions

The isovectors tabulated in the previous section are utilized here to construct the orbital equations [16], which on solving subject to the initial conditions described below, lead us to the similarity solutions. More precisely, we solve

\[
\begin{align*}
\frac{d\bar{t}}{ds} &= \bar{F}', \quad \bar{t}(0) = t, \\
\frac{d\bar{x}}{ds} &= \bar{F}^x, \quad \bar{x}(0) = x, \\
\frac{d\bar{u}}{ds} &= \bar{F}^u, \quad \bar{u}(0) = u,
\end{align*}
\]

(3.1)

where $\bar{F}' = V'(\bar{t}, \bar{x}, \bar{u})$ etc. When the first two of these equations are solved and the parameter $s$ is eliminated between the solutions of these equations, we get the desired similarity variable $\xi = \xi(x, t) = \bar{\xi}(\bar{x}, \bar{t})$. Whereas, eliminating $s$ between the solutions of the third equation and the solution of any one of the former two gives the new dependent variable $F(\bar{\xi}) = \phi(\bar{t}, x, u) = \phi(\bar{t}, \bar{x}, \bar{u})$, which when solved for $u$ gives the required similarity solution, $u = h(x, t)F(\bar{\xi})$, for some function $h$. On substituting this form of $u$ in Eq. (1.1), we arrive at an o.d.e. in the variables $\xi$ and $F(\bar{\xi})$.

It may be mentioned here that for case (i), i.e., $p \neq 1$, we have been able to integrate the corresponding o.d.e.'s for all the cases listed in Table 1. However, the task of obtaining solutions for the o.d.e.'s associated with the cases described in Table 2 has turned out to be difficult one. Consequently, we have attempted to obtain some special solutions. We first proceed to present the results related to case (i).

3.1. Subcase (a) (Table 1, row 1):

On solving the orbital equations corresponding to the case under consideration we arrive at the following form of the similarity solution,

\[
u(x, t) = \left(\frac{3\beta e_1}{\alpha} e^{(\alpha/3\beta)x} + c_2\right)^{-1} F(t), \quad \alpha \neq 0.
\]

(3.2)
<table>
<thead>
<tr>
<th>Case</th>
<th>$\nu'$</th>
<th>$\nu^s$</th>
<th>$\nu^r$</th>
<th>$\nu^y$</th>
<th>$\nu^z$</th>
<th>$\nu^T$</th>
<th>$g(x)$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = p = m = -2$</td>
<td>0</td>
<td>$-\left(\frac{3 \beta c_3}{2} e^{\alpha/3} - c_2\right)$</td>
<td>$-\frac{3 c_1}{(p - 1)} u e^{\alpha/3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$c_3 \left(\frac{3 \beta c_3}{2} e^{\alpha/3} + c_2\right)^{r-1}$</td>
<td>$a = \frac{2x^2}{9p}$, $c_3 &gt; 0$</td>
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<tr>
<td>$q \neq -2$</td>
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<tr>
<td>$3m = p + 2$</td>
<td>0</td>
<td>$-(c_1x + c_2)$</td>
<td>$-\frac{2 c_1}{(p - 1)} u$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$k_0 (c_1x + c_2) \left(1 - \phi_0(p-1)\right)$</td>
<td>$k_0 &gt; 0$</td>
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<tr>
<td>$3n = 2p + 1$</td>
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<tr>
<td>$n \neq p$</td>
<td>0</td>
<td>$-\frac{c_1}{p\rho_0} e^{\alpha/3}$</td>
<td>$-\frac{3 c_1}{(p - 1)} u e^{\alpha/3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$b_1 e^{-3\rho_0} + \psi_0$</td>
<td>$\phi_0 = \frac{\alpha(p - 1)}{3(2p + 1)}$</td>
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<tr>
<td>$n = p = q$</td>
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<tr>
<td>$m = 1$</td>
<td>11p$^2$ + 14p + 2 = 0</td>
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<td></td>
<td>$\nu_0 = \frac{2\mu_0 p(5p + 4)}{3(1 - p)(2p + 1)}$</td>
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<td></td>
<td>$c_1 &gt; 0$</td>
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<tr>
<td>$m = p = q = n$</td>
<td>0</td>
<td>$-\left(\frac{c_1}{\mu_0} e^{\alpha/3} + c_2\right)$</td>
<td>$-\frac{3 c_1}{(p - 1)} u e^{\alpha/3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$c_3 \left(\frac{c_1}{\mu_0} e^{\alpha/3} + c_2\right)^{r-3}$</td>
<td>$a = \frac{\mu_0 \alpha(11p^2 + 14p + 2)}{3(p - 1)(2p + 1)} + \frac{\mu_0^2 \rho_0 p(p + 2)}{(p - 1)^2}$</td>
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\textsuperscript{a} $k_0, b_1, c, \alpha$s are arbitrary constants.
Table 2
Tabulation of the components of the isovector field $V$ for different combinations of the parameters $m, n, p$ and $q$ with $p = 1^a$

<table>
<thead>
<tr>
<th>Case</th>
<th>$V^x$</th>
<th>$V^y$</th>
<th>$V^z$</th>
<th>$V^e$</th>
<th>$V^f$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 2n - 1$</td>
<td>$-(c_1t + c_3)$</td>
<td>$-\frac{1}{3}(c_1x + c_3)$</td>
<td>$\frac{c_1u}{3(n-1)}$</td>
<td>$\frac{n(c_1)}{3(n-1)}v$</td>
<td>$\frac{(2n-1)c_1v}{3(n-1)}$</td>
<td>$c_0(\nu_1 + \nu_0)\frac{c_0}{c_3}, c_0 &gt; 0$</td>
</tr>
<tr>
<td>$n \neq 1$</td>
<td>$n = n = 1$</td>
<td>$-c_2$</td>
<td>$-c_3$</td>
<td>$-c_4$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$q$: arbitrary</td>
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<td></td>
</tr>
</tbody>
</table>

$n = 1, m = 2, q = 0$

(a) $-c_1$ $-c_3$ $-\frac{k_2}{2u}$ $0$ $0$ $0$ $k_1$, constant

(b) $-c_1$ $-\frac{b_1e^{at} + b_2e^{-at}}{2a}$ $-\frac{w}{2a}(b_1e^{at} - b_2e^{-at})$ $0$ $0$ $0$ $k_2 + k_2, k_1 \neq 0$

(c) $-c_1$ $-\frac{k_2}{2u}(-b_1 \sin w + b_2 \cos w)$ $-\frac{w}{2a}(-b_1 \sin w + b_2 \cos w)$ $0$ $0$ $0$ $k_3 + k_2, k_1 \neq 0$

$n = 1$

$m = 2$

$q = 1$

$n = 1$

$m \neq 1, 2$

$q$: arbitrary

\(a\) $w, c_1, b_1, k_2$ are arbitrary constants.
This, along with
\[ g(x) = c_3 \left( \frac{3\beta c_1}{\alpha} e^{(x/3p)} + c_2 \right)^{q-1}, \quad c_3 > 0, \]
when substituted in Eq. (1.1), yields the o.d.e.
\[ \frac{dF}{dt} - c_3 F^q = 0, \quad q \neq -2. \tag{3.3} \]

Eq. (3.3), when solved, gives
\[ F(t) = \left( (1 - q)(c_3 t + c_4) \right)^{1/(1-q)}, \tag{3.4} \]
where \( c_4 \) is a constant of integration and \( q \neq 1 \).

Thus
\[ u(x,t) = \frac{\left( (1 - q)(c_3 t + c_4) \right)^{1/(1-q)}}{\left( 3\beta c_1 \alpha \right) e^{(x/3p)} + c_2}. \tag{3.5} \]

When \( q = 1 \), Eq. (3.3) can be integrated easily and the solution \( u(x,t) \) turns out to be
\[ u(x,t) = \frac{c_4 e^{c_4 t}}{\left( 3\beta c_1 \alpha \right) e^{(x/3p)} + c_2}, \quad c_4 > 0 \text{ a const.} \tag{3.6} \]

3.2. Subcase (b) (Table 1, row 2)

In this case, Eq. (3.1) yield the following.
\[ u(x,t) = (c_1 x + c_2)^{3/(p-1)} F(t), \tag{3.7} \]
where \( 3m = p + 2, \; 3n = 2p + 1 \).

The reduced o.d.e. is
\[ \frac{dF}{dt} + \frac{c_1 (p + 2)}{(p - 1)} \left[ \alpha F^n - \frac{\alpha c_1 (2p + 1)}{(p - 1)} F^n + \frac{3\beta c_1^2 p (2p + 1)}{(p - 1)^2} F^p \right] - k_0 F^q = 0. \tag{3.8} \]

Thus, \( F(t) \) may be obtained from the integral
\[ \int \frac{dF}{r_1 (\alpha F^n - r_2 F^n + r_3 F^p) - k_0 F^q} = -t + C_0, \tag{3.9} \]
where \( C_0 \) denotes an arbitrary constant and
\[ r_1 = \frac{c_1 (p + 2)}{(p - 1)}, \quad r_2 = \frac{\alpha c_1 (2p + 1)}{(p - 1)}, \quad r_3 = \frac{3\beta c_1^2 p (2p + 1)}{(p - 1)^2}. \]
3.3. Subcase (c) (Table 1, row 3)

It is worth recalling for this case that $p$ here stands for any of the two roots of the quadratic equation $11p^2 + 14p + 2 = 0$, and $\mu_0$ and $v_0$ are some definite constants as given under the column of remarks in Table 1. From Eq. (3.1), we have

$$u(x, t) = e^{(3/1)(p-1)x}F(t).$$ \hspace{1cm} (3.10)

The corresponding o.d.e. assumes the following form

$$\frac{dF}{dt} + \frac{3a\mu_0}{(p-1)}F = b_1F^p,$$ \hspace{1cm} (3.11)

which being Bernoulli's equation can be easily solved to derive

$$F(t) = \left\{ b_2e^{\frac{3a\mu_0}{3a\mu_0}} + \frac{b_1(p-1)}{3a\mu_0} \right\}^{1/(1-p)}.$$ \hspace{1cm} (3.12)

Thus, we obtain

$$u(x, t) = e^{(3/1)(p-1)x}\left\{ b_2e^{\frac{3a\mu_0}{3a\mu_0}} + \frac{b_1(p-1)}{3a\mu_0} \right\}^{1/(1-p)}.$$ \hspace{1cm} (3.13)

3.4. Subcase (d) (Table 1, row 4)

This case requires that the coefficient 'a' must be specified as

$$a = \frac{x^2(11p^2 + 14p + 2)}{9\beta(2p + 1)^2}, \hspace{1cm} p \neq -\frac{1}{2}.$$ 

From orbital equations (3.1), we get

$$u(x, t) = \left( \frac{c_1}{\mu_0}e^{\mu_0x} + c_2 \right)^{3/1(p-1)}F(t),$$ \hspace{1cm} (3.14)

which when substituted in Eq. (2.1) gives

$$\frac{dF}{dt} - \left\{ c_3 + \frac{x^2p(p+2)c_2}{9\beta^2(2p+1)^2} \right\}F^p = 0.$$ \hspace{1cm} (3.15)
It further yields

$$F(t) = \left[ (1 - p)(\lambda_0 t + c_4) \right]^{1/(1-p)},$$

(3.16)

where $\lambda_0$ represents the coefficient of $F^p$ in Eq. (3.15) and $c_4$ is a constant. Thus, we arrive at the following expression for $u$:

$$u(x,t) = \left( \frac{c_1}{\mu_k} e^{\mu_k x} + c_2 \right)^{3/(p-1)} \left[ (1 - p)(\lambda_0 t + c_4) \right]^{1/(1-p)}. \quad (3.17)$$

It may be noted that the solution (3.17) for $p = -2$ coincides with the one obtained from Eq. (3.5) under the limit $p(=q) \to -2$.

As stated earlier we now present Table 3 comprising the similarity solution and the associated o.d.e.'s in respect of the various subcases listed in Table 2. Further, we give below some special solutions for the cases of rows 4 and 5.

Note that the o.d.e.'s corresponding to rows 4 and 5 are identical but the forms of their respective solutions in terms of $u$ shall be quite different. The o.d.e.

$$\beta F'' - \alpha F' + 2aFF' - k_1 \xi - k_2 = 0$$

(3.18)

can be integrated once to yield

$$\beta F'' - \alpha F' + aF^2 - \frac{k_1}{2} \xi^2 - k_2 \xi = d_0,$$

(3.19)

where $d_0$ is a constant of integration. For Eq. (3.19), we seek a solution of the form: $F(\xi) = A\xi + B$ for some constants $A$ and $B$. It requires that

$$A = \pm \sqrt{\frac{k_1}{2a}}, \quad B = \pm \frac{k_2}{\sqrt{2ak_1}} \quad \text{and} \quad d_0 = aB^2 - \alpha A.$$  

(3.20)

Consequently the solution to Eq. (1.1) for the cases of row 4 and 5, respectively can be expressed as

$$u(x,t) = \pm \left( \frac{\sqrt{\frac{k_1}{2a}}x + \frac{k_2}{\sqrt{2ak_1}}} + b_1 \left( \frac{1}{2a} \pm \frac{1}{w} \sqrt{\frac{k_1}{2a}} \right) e^{\omega t} + \frac{b_2}{c_1} \left( \frac{1}{2a} \pm \frac{1}{w} \sqrt{\frac{k_1}{2a}} \right) e^{-\omega t} \right)$$

(3.21)

and

$$u(x,t) = \pm \left( \frac{\sqrt{\frac{k_1}{2a}}x + \frac{k_2}{\sqrt{2ak_1}}} + \frac{1}{c_1} \left( \frac{b_2}{2a} \pm \frac{b_1}{v} \sqrt{\frac{k_1}{2a}} \right) \sin \omega t + \frac{1}{c_1} \frac{1}{2a} \pm \frac{b_2}{v} \sqrt{\frac{k_1}{2a}} \cos \omega t \right).$$

(3.22)
### Table 3
Tabulation of similarity variable, similarity solutions and the reduced ordinary differential equations

<table>
<thead>
<tr>
<th>Row of Table 2</th>
<th>$\zeta = \zeta(x,t)$</th>
<th>Similarity solution</th>
<th>The reduced o.d.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{(c_1 x + c_2)}{(c_1 t + c_2)^{1/3}}$</td>
<td>$\nu = (c_1 t + c_2)^{-(3n-1)}F(\xi)$</td>
<td>$\beta c_1^3 F'' - \alpha c_1^2 (F')^n + \alpha c_1 (F^{2n-1})' - \frac{c_1}{3} \left( \frac{F''}{u-1} + \xi F' \right) - c_0 \xi^p F'' = 0$, $c_1 \neq 0$</td>
</tr>
<tr>
<td>2</td>
<td>$x - \left( \frac{c_3}{c_2} \right) t$</td>
<td>$\nu = \exp - \frac{c_3}{c_2} dF(\xi)$</td>
<td>$\beta F'' - \alpha F^n + \left( a - \frac{c_2}{c_3} \right) F' - \frac{c_1}{c_2} F - b_1 \exp \left( \frac{q-1}{c_3} \right) c_3 \xi F' = 0$</td>
</tr>
<tr>
<td>3</td>
<td>$x - \frac{1}{c_1} \left( \frac{k_1}{2} t^2 + k_2 t \right)$</td>
<td>$u = \frac{k_1}{2ac_1} t + F(\xi)$</td>
<td>$\beta F'' - \alpha F^n + 2\alpha F' F' - \frac{k_4}{c_1} F' - \left( k_1 + \frac{k_2}{2ac_1} \right) = 0$</td>
</tr>
<tr>
<td>4</td>
<td>$x - \frac{1}{c_1} \left( b_1 e^{\alpha t} - b_2 e^{-\alpha t} \right)$</td>
<td>$u = \left( b_1 e^{\alpha t} + b_2 e^{-\alpha t} \right) + F(\xi)$</td>
<td>$\beta F'' - \alpha F^n \pm 2\alpha F' - \left( k_1 \xi + k_2 \right) = 0$</td>
</tr>
<tr>
<td>5</td>
<td>$x - \frac{1}{c_1} \left( b_1 \sin vt - b_2 \cos vt \right)$</td>
<td>$u = \left( b_1 \cos vt + b_2 \sin vt \right) + F(\xi)$</td>
<td>$\beta F'' - \alpha F^n \pm 2\alpha F' - \left( k_1 \xi + k_2 \right) = 0$</td>
</tr>
<tr>
<td>6</td>
<td>$x - \frac{1}{c_1} \left( \frac{k_1}{k_2} e^{\alpha t} + k_2 t \right)$</td>
<td>$\nu = \frac{k_1}{2ac_1} e^{\alpha t} + F(\xi)$</td>
<td>$\beta F'' - \alpha F^n \pm 2\alpha F' - k_1 F' - k_1 \frac{k_4}{2ac_1} = 0$</td>
</tr>
<tr>
<td>7</td>
<td>$x - \left( \frac{c_3}{c_2} \right) t$</td>
<td>$\nu = F(\xi)$</td>
<td>$\beta F'' - \alpha F^n \pm \alpha F'^{-1} + F' - k_1 F' - \frac{c_1}{c_2} F'' = 0$</td>
</tr>
</tbody>
</table>
4. Concluding remarks

The study carried out in this paper through an alternative method of isovectors to obtain the continuous group of invariance transformations has not only provided some interesting results but also confirms the efficacy and the potential of the method, which though still remains much to be explored from applications point of view, over other conventional methods of obtaining the symmetry groups. We have indeed planned to highlight this feature of the method in our future communication.

It can be remarked that the kind of relationships which the method furnishes among the various parameters of the equation and the form of the arbitrary function are not easily conceivable otherwise, (see for example – column of remarks of Table 1).

Another remarkable outcome of the approach, in the case of $p \neq 1$, has been the fact that the resulting o.d.e's in each subcase being of first order turns out to be completely integrable leading eventually to exact closed form solutions.

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References