On Incomplete Lagrange Function and Saddle Point
Optimality Criteria in Mathematical Programming

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An incomplete Lagrange function is used to study saddle point optimality criteria
for a class of nonlinear programming problems under generalized convexity as-
sumptions. This study is further extended to certain class of fractional and
generalized fractional programming problems as well.

1. INTRODUCTION

Let $R^n$ denote the $n$-dimensional Euclidean space and let $R^n_+$ be its
non-negative orthant. Let $f: R^n \to R$ and $h_j: R^n \to R$ ($j = 1, 2, \ldots, m$).
We now consider the following nonlinear programming problem:

\[
\begin{align*}
\text{(P)} \quad & \quad \text{Min } f(x) \\
\text{subject to } & \quad h_j(x) \leq 0 \quad (j = 1, 2, \ldots, m).
\end{align*}
\]

For $M = \{1, 2, \ldots, m\}$, $J \subseteq M$, let $K = M \setminus J$ be the set of indices $i$
which are in $M$ but not in $J$. Let $h(x)$ denote the column vector
$(h_1(x), h_2(x), \ldots, h_m(x))^T$ and be partitioned as $h(x) = (h_J(x), h_K(x))^T$. 
Further, let \( X = \{ x \in \mathbb{R}^n : h_k(x) \leq 0, \ k \in K \} \), and \(|J|\) and \(|K|\) denote the number of elements in the subsets \( J \) and \( K \), respectively. For \( \lambda_j \in \mathbb{R}_+^{\|J\|} \) the Lagrangian dual of problem (P) is defined as

\[
(\text{LD}) \quad \max_{\lambda_j \geq 0} \min_{x \in X} \left[ f(x) + (\lambda_j)^T h_j(x) \right].
\] (2)

In the usual Lagrangian duality [11], the set \( K \) is normally taken to be an empty set, a consequence of which is that \( X \) becomes \( \mathbb{R}^n \). However, in general the set \( K \) may not be empty. To emphasize this fact we call (LD) the mixed Lagrangian dual and the function \( L : X \times \mathbb{R}_+^{\|J\|} \rightarrow \mathbb{R} \), given by

\[
L(x, \lambda_j) = f(x) + (\lambda_j)^T h_j(x),
\] (3)

the incomplete Lagrange function of the primal problem (P).

It is well known [2] that various duality theorems hold between (P) and (LD) under appropriate convexity type assumptions, on the objective and the constraint functions. Also under the same convexity assumptions, saddle point optimality criteria are also available in terms of a saddle point of the incomplete Lagrange function \( L(x, \lambda_j) \). However, it may be noted here that there is no partitioning of constraints in the problem studied by Bazarra et al. [2] and thus the set \( \mathcal{X} \) of [2] need not be obtained through the partitioning constraints \( h_k (k \in K) \).

In order to have a deeper insight of the Mond–Weir type dual [10], Bector et al. [4] recently studied the incomplete Lagrange function \( L(x, \lambda_j) \) and its associated Lagrangian dual (LD) from a different point of view. The main observation of [4] is the fact that the Mond–Weir type dual is connected with the incomplete Lagrange function \( L(x, \lambda_j) \) exactly in the same manner as the usual Lagrange function

\[
L(x, \lambda) = f(x) + \lambda^T h(x)
\]

is connected to the Wolfe dual.

The purpose of the present paper is to study saddle point properties of the incomplete Lagrange function and relate them to the optimal solutions of certain nonlinear programming problems, in particular, for fractional and generalized fractional programming problems. This will, of course, require choosing a suitable incomplete Lagrange function for a given class of problems.

In what follows we shall assume that \( f \) and \( h_j (j = 1, 2, \ldots, m) \) are twice differentiable and the symbols \( \nabla \) and \( \nabla^2 \) denote the gradient and the Hessian operators, respectively. Also, the vector \( \lambda \in \mathbb{R}_+^n \) will be written as \( \lambda = (\lambda_j, \lambda_K) \), \( \lambda_j \in \mathbb{R}_+^{\|J\|} \) and \( \lambda_K \in \mathbb{R}_+^{\|K\|} \). Any additional assumption will be made as and when needed.
2. INCOMPLETE LAGRANGE FUNCTION AND SADDLE POINT OPTIMALITY CRITERIA

This section is divided into three subsections. In Section 2.1 we study incomplete Lagrange function and saddle point optimality criteria in nonlinear programming, whereas Sections 2.2 and 2.3 are devoted to the extension of these results to certain fractional and generalized fractional programming problems by choosing suitable incomplete Lagrange functions. We need the following definitions in the sequel [7, 8]:

**Definition 2.1.** A differentiable function \( g: S \rightarrow R \), where \( S \subseteq R^n \), is said to be \textit{invex} with respect to the function \( \eta: S \times S \rightarrow R^n \), if for all \( x, u \in S \)
\[
g(x) - g(u) \geq \eta(x, u)^T \nabla g(u).
\]

**Definition 2.2.** A differentiable function \( g: S \rightarrow R \), where \( S \subseteq R^n \), is said to be \textit{pseudo invex} with respect to the function \( \eta: S \times S \rightarrow R^n \), if for all \( x, u \in S \)
\[
\eta(x, u)^T \nabla g(u) \geq 0 \quad \Rightarrow \quad g(x) \geq g(u).
\]

**Definition 2.3.** A differentiable function \( g: S \rightarrow R \), where \( S \subseteq R^n \), is said to be \textit{quasi invex} with respect to the function \( \eta: S \times S \rightarrow R^n \), if for all \( x, u \in S \)
\[
g(x) \leq g(u) \quad \Rightarrow \quad \eta(x, u)^T \nabla g(u) \leq 0.
\]

2.1. Nonlinear Programming

Consider the nonlinear programming problem \((P)\) and its associated incomplete Lagrange function \( L: X \times R^{|J|} \rightarrow R \) given by
\[
L(x, \lambda_j) = f(x) + \lambda_j^T h_j(x).
\]
We now have the following definition:

**Definition 2.4.** A point \( (\tilde{x}, \lambda_j) \in X \times R^{|J|} \) is called a \textit{saddle point} of the incomplete Lagrange function \( L \) if
\[
L(\tilde{x}, \lambda_j) \leq L(x, \lambda_j) \leq L(x, \lambda_j),
\]
for all \( x \in X \) and \( \lambda_j \in R^{|J|} \).

**Theorem 2.1 (necessary condition).** Let, for all fixed \( \lambda_j \in R^{|J|} \) and \( \lambda_k \in R^{|K|} \), \( f(\cdot) + (\lambda_j)^T h_j(\cdot) \) be pseudo invex with respect to \( \eta \) and \( (\lambda_k)^T h_k(\cdot) \) be quasi invex with respect to the same \( \eta \). Let \( \tilde{x} \) be optimal for \((P)\) at which a suitable constraint qualification [9] holds. Then there exists \( \lambda_j \in R^{|J|} \) such that \((\tilde{x}, \lambda_j)\) is a saddle point of the incomplete Lagrange function \( L \).
Proof. Since \( \bar{x} \) is optimal for (P) at which a constraint qualification [9] holds, there exists \( \bar{\lambda} = (\bar{\lambda}_f, \bar{\lambda}_h) \in R^m, \bar{\lambda}_f \in R^{|J_f|}, \bar{\lambda}_h \in R^{|K|} \), such that

\[
\nabla \left[ f(\bar{x}) + (\bar{\lambda}_f)^T h_f(\bar{x}) + (\bar{\lambda}_h)^T h_h(\bar{x}) \right] = 0, \tag{4}
\]

\[
(\bar{\lambda}_f)^T h_f(\bar{x}) = 0, \tag{5}
\]

\[
(\bar{\lambda}_h)^T h_h(\bar{x}) = 0, \tag{6}
\]

\[
\bar{\lambda} = (\bar{\lambda}_f, \bar{\lambda}_h) \geq 0. \tag{7}
\]

Let \( x \in X; \) then \( (\bar{\lambda}_h)^T h_h(x) \leq 0 \) and therefore from (5) we have

\[
(\bar{\lambda}_h)^T h_h(\bar{x}) \leq (\bar{\lambda}_h)^T h_h(x), \tag{8}
\]

which, by the assumption of the quasi invexity of the function \( (\bar{\lambda}_h)^T h_h(\cdot) \), yields

\[
\eta^T (x, \bar{x}) \nabla \left[ (\bar{\lambda}_h)^T h_h(\bar{x}) \right] \leq 0. \tag{9}
\]

Now from (4) and (9) we have

\[
\eta^T (x, \bar{x}) \nabla \left[ f(\bar{x}) + (\bar{\lambda}_f)^T h_f(\bar{x}) \right] \geq 0. \tag{10}
\]

Therefore by the pseudo invexity of the function \( f(\cdot) + (\bar{\lambda}_f)^T h_f(\cdot) \) we have

\[
f(x) + (\bar{\lambda}_f)^T h_f(x) \geq f(\bar{x}) + (\bar{\lambda}_f)^T h_f(\bar{x});
\]

i.e.,

\[
L(\bar{x}, \bar{\lambda}_f) \leq L(x, \bar{\lambda}_f), \quad \forall x \in X. \tag{11}
\]

Again, let \( \lambda_f \in R^{|J_f|} \). Therefore

\[
(\lambda_f)^T h_f(\bar{x}) \leq 0,
\]

and hence from (6) we get

\[
f(\bar{x}) + (\lambda_f)^T h_f(\bar{x}) \leq f(\bar{x}) + (\bar{\lambda}_f)^T h_f(\bar{x});
\]

i.e.,

\[
L(\bar{x}, \lambda_f) \leq L(\bar{x}, \bar{\lambda}_f), \quad \text{for all } \lambda_f \in R^{|J_f|} \tag{12}
\]

On combining (11) and (12) we get

\[
L(\bar{x}, \lambda_f) \leq L(\bar{x}, \bar{\lambda}_f) \leq L(x, \bar{\lambda}_f)
\]

for all \( x \in X \) and \( \lambda_f \in R^{|J_f|} \).
THEOREM 2.2. Let \((\bar{x}, \bar{\lambda})\) be a saddle point of the incomplete Lagrange function \(L(x, \lambda_j)\). Then \((\bar{\lambda}_j)^T h_j(\bar{x}) = 0\) and \(\bar{x}\) is optimal for \((P)\).

Proof. The proof is similar to that of [9, Theorem 5.3.1; 1, Theorem 3.12].

2.2. Fractional Programming Problem

We now consider the fractional programming problem

\[
\begin{align*}
\text{(FP)} & \quad \text{Min} & & \left( \frac{f(x)}{g(x)} \right) \\
& \text{subject to} & & h_j(x) \leq 0, \quad (j = 1, 2, \ldots, m),
\end{align*}
\]

where the set \(S = \{x \in \mathbb{R}^n : h_j(x) \leq 0, \quad j = 1, 2, \ldots, m\}\) is the set of feasible solutions and \(f, g, h_j : \mathbb{R}^n \to \mathbb{R}, \quad j \in M\) are twice differentiable functions with \(g(x) > 0\) for all \(x \in S\). Since \(g(x) > 0\) for all \(x \in S\), the problem (FP) is equivalent to the problem

\[
\begin{align*}
\text{(FP1)} & \quad \text{Min} & & \left( \frac{f(x)}{g(x)} \right) \\
& \text{subject to} & & h_j(x)/g(x) \leq 0, \quad j \in J, \\
& & & h_k(x) \leq 0, \quad k \in K.
\end{align*}
\]

This form of (FP1) suggests the choice of the incomplete Lagrange function \(L_F : X \times R^{|J|}_+ \to R\) as

\[
L_F(x, \lambda_j) = \left( \frac{f(x) + (\lambda_j)^T h_j(x)}{g(x)} \right),
\]

where \(X = \{x \in \mathbb{R}^n : h_k(x) \leq 0, \quad k \in K\}\). It may be observed here that the Lagrange function \(L_F\) is different from the one considered by Bector [3].

We now have the following definition:

DEFINITION 2.5. A point \((\bar{x}, \bar{\lambda}_j) \in X \times R^{|J|}_+\) is called a saddle point of the incomplete Lagrange function \(L_F\) if

\[
L_F(\bar{x}, \lambda_j) \leq L_F(x, \lambda_j) \leq L_F(\bar{x}, \bar{\lambda}_j),
\]

for all \(x \in X\) and \(\lambda_j \in R^{|J|}_+\).
THEOREM 2.3 (necessary condition). Let, for all fixed \( \lambda_f \in R^f_+ \) and \( \lambda_K \in R^K_+ \), \( (f(\cdot) + (\lambda_f)^T h_f(\cdot))/g(\cdot) \) be pseudo invex with respect to \( \eta \) and let \( (\lambda_K)^T h_K(\cdot) \) and a suitable constraint qualification \([9]\) holds for (FP1). Then there exists \( \lambda_f \in R^f_+ \) such that \((\bar{x}, \lambda_f)\) is a saddle point of the incomplete Lagrange function \( L_F \).

Proof. Since \( \bar{x} \) is an optimal solution for (FP) (and hence for (FP1)) and a suitable constraint qualification \([9]\) holds for (FP1), there exists \( \lambda^* = (\lambda_f^*, \lambda_K^*) \in R^m, \lambda_f^* \in R^f_+, \lambda_K^* \in R^K_+ \), such that

\[
\begin{align*}
\nabla \left[ \frac{f(\bar{x})}{g(\bar{x})} + (\lambda_f^*)^T h_f(\bar{x}) + (\lambda_K^*)^T h_K(\bar{x}) \right] &= 0, \\
(\lambda_f^*)^T h_f(\bar{x}) &= 0, \\
(\lambda_K^*)^T h_K(\bar{x}) &= 0, \\
\lambda^* &= (\lambda_f^*, \lambda_K^*) \geq 0.
\end{align*}
\]

Now following the lines of the proof of Theorem 2.1, we obtain

\[
\eta^T(x, \bar{x}) \nabla \left[ \frac{f(\bar{x})}{g(\bar{x})} + (\lambda_f^*)^T h_f(\bar{x}) \right] \geq 0, \quad (13)
\]

for all \( x \in X \). Therefore by setting \( \bar{\lambda}_f = \lambda_f^* g(\bar{x}) \in R^f_+ \), from (13) we get

\[
\eta^T(x, \bar{x}) \nabla \left[ \frac{f(\bar{x})}{g(\bar{x})} + \frac{(\bar{\lambda}_f)^T h_f(\bar{x})}{g(\bar{x})} \right] \geq 0.
\]

The remaining part of the proof follows on the lines of the proof of Theorem 2.1.

THEOREM 2.4 (sufficient condition). Let \((\bar{x}, \bar{\lambda}_f)\) be a saddle point of the incomplete Lagrange function \( L_F \); then \( \bar{\lambda}_f^T h_f(\bar{x}) = 0 \) and \( \bar{x} \) is optimal for (FP).

Proof. The proof follows on the lines of \([9, \text{Theorem } 5.31; 1, \text{Theorem } 3.12]\).

Remark 2.1. For the usual convex/concave fractional programming, \( f, -g, \) and \( h \) are convex functions. In case \( g \) is not an affine function, it will have to be assumed that \( f(x) \geq 0 \) for all \( x \in R^m \) and this will necessitate taking \( f(u) + (\lambda_f)^T h_f(u) \geq 0 \) in the statement of Theorem 2.3. These assumptions are also required when \( f, -g, \) and \( h \) are invex functions.
2.3. *Generalized Fractional Programming Problem*

Let \( f_i, g_i : \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, 2, \ldots, p \), and \( h_j : \mathbb{R}^n \to \mathbb{R} \) for \( j = 1, 2, \ldots, m \) be continuous and twice differentiable functions. We consider the generalized fractional programming problem

\[
\text{(GFP)} \quad \begin{array}{c}
\text{Min} \\
\text{subject to}
\end{array} \quad \begin{array}{c}
\text{Max} \quad \frac{f_i(x)}{g_i(x)} \\
\end{array} \\
\begin{array}{c}
\forall x \in \mathbb{R}^n, \quad \sum_{i=1}^{p} \lambda_i \leq p
\end{array}
\]

where \( \bar{S} = \{ x \in \mathbb{R}^n : h_j(x) \leq 0, \quad j = 1, 2, \ldots, m \} \) is the set of feasible solutions and \( g_i(x) > 0 \) for all \( x \in \bar{S} \) and for all \( i = 1, 2, \ldots, p \). The incomplete Lagrange function \( L_G : \mathbb{R}^n \to \mathbb{R} \) for the problem (GFP) can be chosen as

\[
L_G(x, y, \lambda) = \frac{y^T f(x) + (\lambda_j)^T h_j(x)}{y^T g(x)}
\]

(14)

where \( Y = \{ y \in \mathbb{R}^n \} \) and \( f(x) \) denotes the column vector.

\([f_1(x), f_2(x), \ldots, f_p(x)]^T\), and \( g(x) \), and \( h(x) \) are defined similarly with appropriate number of components. It may be observed here that the incomplete Lagrange function \( L_G \) is different from the one considered by Bector and Suneja [6]. We now have the following definition:

**Definition 2.6.** A point \((\bar{x}, \bar{y}, \bar{\lambda}) \in \bar{S} \) is said to be the saddle point of the incomplete Lagrange function \( L_G \) if

\[
L_G(x, y, \lambda) \leq L_G(\bar{x}, \bar{y}, \bar{\lambda}) \leq L_G(x, y, \lambda)
\]

for all \( x \in \bar{S}, y \in Y \), and \( \lambda \in \mathbb{R}^p \).

To establish the saddle point optimality criteria for the problem (GFP) we consider the following parametric program in the parameter \( v \):

\[
\text{(GFP)}_v \quad F(v) = \text{Min} \quad \begin{array}{c}
\text{Max} \quad \frac{f_i(x)}{g_i(x)} \\
\forall x \in \mathbb{R}^n, \quad \sum_{i=1}^{p} \lambda_i \leq p
\end{array}
\]

Now (GFP)\(_v\) is equivalent to the following nonlinear programming problem (EGFP)\(_v\):

\[
\text{(EGFP)}_v \quad \begin{array}{c}
\text{Min} \quad q \\
\text{subject to}
\end{array} \quad \begin{array}{c}
\forall x \in \mathbb{R}^n, \quad \sum_{i=1}^{p} \lambda_i \leq p \\
\end{array}
\]

\[
\begin{array}{c}
f_i(x) - v g_i(x) \leq q \\
h_j(x) \leq 0
\end{array} \quad \begin{array}{c}
(i = 1, 2, \ldots, p), \\
(j = 1, 2, \ldots, m).
\end{array}
\]
We now have the following lemmas [5]:

**Lemma 2.1.** If (GFP) has an optimal solution \( x^* \) with optimal value of the (GFP) objective as \( v^* \), then \( F(v^*) = 0 \); conversely, if \( F(v^*) = 0 \), then (GFP) and (GFP)_\( v \) have the same set of optimal solutions.

**Lemma 2.2.** If \((x, v, q)\) is (EGFP)_\( v \) feasible, then \( x \) is (GFP) feasible. If \( x \) is (GFP) feasible, then there exists \( v \) and \( q \) such that \((x, v, q)\) is (EGFP)_\( v \) feasible.

**Lemma 2.3.** The point \( x^* \) is (GFP) optimal with corresponding optimal value of the (GFP) objective equal to \( v^* \) iff \((x^*, v^*, q^*)\) is (EGFP)_\( v \) optimal with corresponding optimal value of the (EGFP)_\( v \) objective equal to zero; i.e., \( q^* = 0 \).

**Theorem 2.5** (necessary condition). Let for all fixed \( y \in Y, \lambda \in R^{|\mathcal{I}|}_+ \), and \( \lambda_\mathcal{K} \in R^{|\mathcal{K}|}_+ \), let \((y^T f(\cdot) + (\lambda_j)^T h_j(\cdot))/y^T g(\cdot)\) be pseudo invex with respect to \( \eta \) and let \((\lambda_k)^T h_k(\cdot)\) be quasi invex with respect to the same \( \eta \). Let \( \bar{x} \) be optimal for (GFP) and a suitable constraint qualification [9] holds for (EGFP)_\( v \). Then there exists \((\bar{y}, \bar{\lambda}, \bar{\lambda}_\mathcal{K}) \in Y \times R^{|\mathcal{I}|}_+ \) such that \((\bar{x}, \bar{y}, \bar{\lambda}_\mathcal{K})\) is a saddle point for the incomplete Lagrange function \( L_G \).

**Proof.** Corresponding to (GFP)-optimal solution \( \bar{x} \), let the optimal value of the (GFP)-objective be denoted by \( \bar{v} \). Then, by Lemma 2.3, \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{\lambda}_\mathcal{K})\) is an optimal solution to the problem (EGFP)_\( v \) with \( \bar{q} = 0 \). Therefore, in view of the constraint qualification, there exists \((\bar{y}, \bar{\lambda}_\mathcal{I}) \in Y \times R^{|\mathcal{I}|}_+ \) such that

\[
\nabla \left[ \bar{y}^T f(\bar{x}) - \bar{y}^T g(\bar{x}) + (\bar{\lambda}_\mathcal{I})^T h_\mathcal{I}(\bar{x}) + (\bar{\lambda}_\mathcal{K})^T h_\mathcal{K}(\bar{x}) \right] = 0, \tag{15}
\]

\[
\bar{y}_i [f_i(\bar{x}) - \bar{g}_i(\bar{x})] = 0 \quad (i = 1, 2, \ldots, p), \tag{16}
\]

\[
(\bar{\lambda}_\mathcal{I})^T h_\mathcal{I}(\bar{x}) = 0, \tag{17}
\]

\[
(\bar{\lambda}_\mathcal{K})^T h_\mathcal{K}(\bar{x}) = 0, \tag{18}
\]

\[
\sum_{i=1}^{p} \bar{y}_i = 1, \tag{19}
\]

\[
\bar{\lambda} = (\bar{\lambda}_\mathcal{I}, \bar{\lambda}_\mathcal{K}) \geq 0, \tag{20}
\]

\[
\bar{\eta} \geq 0. \tag{21}
\]

Let \( x \in X \); then \( \bar{\lambda}_\mathcal{K}^T h_\mathcal{K}(x) \leq 0 \) and therefore from (18) we have

\[
(\bar{\lambda}_\mathcal{K})^T h_\mathcal{K}(x) \leq \bar{\lambda}_\mathcal{K}^T h_\mathcal{K}(\bar{x}),
\]
which by the hypotheses of the theorem gives
\[ \eta^T(x, \tilde{x}) \nabla \left[ \tilde{\lambda}_k^T h_k(\tilde{x}) \right] \leq 0 \quad \forall x \in X. \quad (22) \]

From (15) and (22) we obtain
\[ \eta^T(x, \tilde{x}) \nabla \left[ \tilde{y}^T f(\tilde{x}) - \tilde{\nu}^T g(\tilde{x}) + (\tilde{\lambda}_T)^T h_T(\tilde{x}) \right] \geq 0 \quad \forall x \in X. \quad (23) \]

Now from (16) and (17) we have
\[ \tilde{\nu} = \frac{\tilde{y}^T f(\tilde{x})}{\tilde{y}^T g(\tilde{x})} = \frac{\tilde{y}^T f(\tilde{x}) + \tilde{\lambda}_T^T h_T(\tilde{x})}{\tilde{y}^T g(\tilde{x})}. \quad (24) \]

Substituting this value of \( \tilde{\nu} \) in (23) and using (19), we get for all \( x \in X \)
\[ \eta^T(x, \tilde{x}) \left[ \nabla \left( \tilde{y}^T f(\tilde{x}) + (\tilde{\lambda}_T)^T h_T(\tilde{x}) \right) \right. \]
\[ - \left. \left( \frac{\tilde{y}^T f(\tilde{x}) + (\tilde{\lambda}_T)^T h_T(\tilde{x})}{\tilde{y}^T g(\tilde{x})} \right) \nabla \tilde{y}^T g(\tilde{x}) \right] \geq 0. \]

This is equivalent to
\[ \eta^T(x, \tilde{x})^T \nabla \left( \frac{\tilde{y}^T f(\tilde{x}) + (\tilde{\lambda}_T)^T h_T(\tilde{x})}{\tilde{y}^T g(\tilde{x})} \right) \geq 0, \quad \forall x \in X, \quad (25) \]

which along with the hypothesis of the theorem yields
\[ \frac{\tilde{y}^T f(x) + (\tilde{\lambda}_T)^T h_T(x)}{\tilde{y}^T g(x)} \geq \frac{\tilde{y}^T f(\tilde{x}) + (\tilde{\lambda}_T)^T h_T(\tilde{x})}{\tilde{y}^T g(\tilde{x})}, \quad \forall x \in X; \]

i.e.,
\[ L_\phi(x, \tilde{y}, \tilde{\lambda}_T) \geq L_\phi(\tilde{x}, \tilde{y}, \tilde{\lambda}_T) \quad \forall x \in X. \quad (26) \]

Since \( \tilde{x} \) is (GFP)-optimal with \( \tilde{\nu} \) as optimal value, from Lemma 2.2 we have
\[ f_i(\tilde{x}) - \tilde{\nu} g_i(\tilde{x}) \leq \tilde{q} = 0. \quad (27) \]
For \( y \in Y, (27), (16), \) and (19) yield
\[
\frac{y^T f(\tilde{x})}{y^T g(\tilde{x})} \leq \tilde{\nu} = \frac{\tilde{y}^T f(\tilde{x})}{\tilde{y}^T g(\tilde{x})}.
\]

Now let \( \lambda_j^* \in R^{|I|}_v \) and therefore \( \lambda_j^* h_j(\tilde{x}) \leq 0 \). Then from the above inequality we have
\[
\frac{y^T f(\tilde{x})}{y^T g(\tilde{x})} + \lambda_j^* h_j(\tilde{x}) \leq \tilde{\nu}.
\] (28)

Let \( \lambda_j = \lambda_j/y^T g(\tilde{x}) \) for \( \lambda_j \in R^{|I|}_v \); then from (28) we obtain
\[
\frac{y^T f(\tilde{x})}{y^T g(\tilde{x})} + (\lambda_j)^T h_j(\tilde{x}) \leq \tilde{\nu}
\] (29)

and hence
\[
\frac{y^T f(\tilde{x}) + (\lambda_j)^T h_j(\tilde{x})}{y^T g(\tilde{x})} \leq \tilde{\nu} = \frac{\tilde{y}^T f(\tilde{x}) + (\tilde{\lambda}_j)^T h_j(\tilde{x})}{\tilde{y}^T g(\tilde{x})};
\]
i.e.,
\[
L_G(\tilde{x}, y, \lambda_j) \leq L_G(\tilde{x}, \tilde{y}, \tilde{\lambda}_j)
\] (30)

for all \( y \in Y \) and \( \lambda_j \in R^{|I|}_v \).

From (26) and (30) we get
\[
L_G(\tilde{x}, y, \lambda_j) \leq L_G(\tilde{x}, \tilde{y}, \tilde{\lambda}_j) \leq L_G(x, \tilde{y}, \tilde{\lambda}_j)
\]
for all \( x \in S, y \in Y, \) and \( \lambda_j \in R^{|I|}_v \).

**Theorem 2.6 (necessary condition).** Let for all fixed \( y \in Y, \lambda_j \in R^{|I|}_v \), \( v \in R, \) and \( \lambda_j \in R^{|I|}_v \), let \( (y^T f(\cdot) - vy^T g(\cdot) + (\lambda_j)^T h_j(\cdot)) \) be invex with respect to \( \eta \), and let \( (\lambda_j)^T h_j(\cdot) \) be quasi invex with respect to the same \( \eta \). Let \( \tilde{x} \) be optimal for (GFP) and a suitable constraint qualification [9] holds for (EGFP). Then there exists \( (\tilde{y}, \tilde{\lambda}_j) \in Y \times R^{|I|}_v \) such that \( (\tilde{x}, \tilde{y}, \tilde{\lambda}_j) \) is a saddle point for the incomplete Lagrange function \( L_G \).

**Proof.** The proof follows on the lines of Theorem 2.5.

**Theorem 2.7 (sufficient condition).** Let \( (\tilde{x}, \tilde{y}, \tilde{\lambda}_j) \) be a saddle point of the incomplete Lagrange function \( L_G \); then \( (\lambda_j)^T h_j(\tilde{x}) = 0 \) and \( \tilde{x} \) is an optimal solution for (GFP).

**Proof.** The proof follows on the lines of [3].
Remark 2.2. For the case when $f_i$, $-g_i$, and $h_i$ are convex functions and the same $g_i$ is not affine, it will be necessary to assume that $f_i \geq 0$ for all $x \in \mathbb{R}^n$. It will be further required that $(y^T f(u) + (\lambda_j)^T h_j(u))/y^T g(u) \geq 0$ is taken in Theorem 2.5 and $\nu \geq 0$ is taken in Theorem 2.6. These assumptions are also required when $f_i$, $-g_i$, and $h_i$ are invex functions.

Remark 2.3. The results proved here can also be proved, under appropriate conditions and modifications in proofs, for multiobjective, continuous, and variational type programming problems.

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REFERENCES