

Necessity of (5) is proved by noting that stability of $(A + BF)$ implies

$$X(A + BF) = 0, \tag{13}$$

and (5) then follows from (10).

The result of Theorem 1 is obtained directly, since

$$N(C) \subset v \subset N(F) \tag{14}$$

implies a solution K to $F = KC$.

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On the Stability of a Fixed Lag Smoother

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Abstract—The aim of this correspondence is to show that the fixed lag smoothing algorithm reported earlier by the authors is asymptotically stable and is computationally better than the other stable smoothers reported recently.

The first fixed lag smoothing algorithm was developed by Rauch [1] as an extension to the fixed interval smoothing algorithm 121. Meditch [3] rederived the results of [1] using orthogonal projection. He conjectured that the fixed lag smoother [1], [3] would be asymptotically stable under the usual conditions which guarantee asymptotic stability of the Kalman filter. However, Kailath and Frost [4], while deriving the continuous-time form of the smoothing equations, pointed out that the equations representing the smoother and filter form a pair of adjoint systems. This would imply that the above conjecture about the stability of the fixed lag smoothing algorithm is false, as has been established by Kelly and Anderson [5]. These authors have shown that the fixed lag smoothing algorithm in [1], [3], and [4] are unstable, precisely when the corresponding filter is asymptotically stable.

Because of the usefulness of the form of smoothing in providing an on-line estimate that is superior to the filtered estimate, it is desirable to explore the possibility of deriving stable solutions of the smoothing problem. Two such solutions have recently been proposed [6], [7]. The purpose of this note is to show that the smoothing algorithm developed earlier by the present authors [8] is asymptotically stable as long as the corresponding filter is asymptotically stable. The stability is guaranteed by the fact that the structure of the smoother developed is analogous to that of the filter itself. For the sake of completeness, the smoother is derived here for the case of white Gaussian noise.

The system is represented by the pair of equations

$$x_{k+1} = A_k x_k + B_k w_k \tag{1}$$

$$y_k = H_k x_k + v_k \tag{2}$$

where w_k and v_k are mutually independent, white, Gaussian noise sequences with covariances Q_k and R_k , respectively. The objective is to get at the estimate $\hat{x}_{k|k}$ of the state x_k (for fixed X). As a

first step, an augmented vector $Xr' = [Z^{-1}ZE^{-1} \dots Z^{-1}X]$ is formed which obviously follows the propagation and observation models

$$\hat{x}_{k+1} = \begin{bmatrix} A_k & 0 & \dots & 0 & 0 \\ I & 0 & & \cdot & \cdot \\ 0 & I & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \cdot & 0 \end{bmatrix} X_k + \begin{bmatrix} B_k \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} w_k \tag{3}$$

$$y_k = [H_k \ 0 \ \dots \ 0] X_k + v_k \tag{4}$$

Next, from the well-known results of Kalman, the filtered estimate $\hat{x}_{k|k}$ of X_k is obtained as

$$\dot{\hat{x}}_{k|k} = \begin{bmatrix} (I - K_k A_k) A_k & 0 & \dots & 0 & 0 \\ I - K_k H_k & 0 & & \cdot & \cdot \\ -K_k^2 H_k A_k & I & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -K_k^N H_k A_k & 0 & \dots & I & 0 \end{bmatrix} \hat{x}_{k-1|k-1}$$

$$+ \begin{bmatrix} K_k^0 \\ K_k^1 \\ \cdot \\ \cdot \\ K_k^N \end{bmatrix} y_k \tag{5}$$

where K_k^0 is the filtering gain and $K_k^1, K_k^2, \dots, K_k^N$ are the smoothing gains [6] given by

$$K_k^i = P_{k-1}^{-1} (H_k^i P_{k-1} H_k^{i'} + R_k)^{-1} \tag{6}$$

The desired smoothed state $\hat{x}_{k|k}$ is then given by

$$\hat{x}_{k|k} = [0 \dots 0 \ \hat{I}] \hat{S}_{k|k} \tag{7}$$

This shows that the smoother is asymptotically stable if the filter for the augmented system is so. To check the stability of the augmented filter, we consider the homogeneous part of (5) and find its eigenvalues. Note that the characteristic equation has the form

$$\begin{vmatrix} I - (I - K_k^0 H_k) A_k & 0 & \dots & 0 & \cdot \\ K_k^1 H_k & -I & & Xr & \cdot \\ K_k^2 H_k A_k & -I & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_k^N H_k A_k & 0 & \dots & -I & \lambda I \end{vmatrix} = 0. \tag{8}$$

Alternatively, expanding the determinant about its elements of the first row, we get

$$X^n |IXI - (I - K_k^0 H_k) A_k| = 0 \tag{9}$$

where n is the dimension of the state x_k .

Apparently then, nN of the eigenvalues of the augmented filter are located at the origin of the s -plane, whereas the remaining n eigenvalues are the same as those of the Kalman filter for the original system (1), (2). It therefore follows that the smoothing algorithm (S)-(i) is asymptotically stable whenever the Kalman filter for (1), (3) is asymptotically stable.

It may be of interest to note that the proposed fixed lag algorithm has been employed by the authors for obtaining smoothed state estimate of a number of linear and nonlinear, discrete-time as well as continuous-time systems. Some of these results will be reported

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elsewhere. It has been shown [9] that this particular algorithm involves $(9 - 1)n^3 + (2n + 2 + m)nmN - mn(n + 1)/2$ major computer operations at each stage (m represents dimension of observation vector). For small values of n , this compares with $(3.5 - V + 2)n^3 + (1.5 - V + 1)n^3$ major operations involved for the algorithm in [SI]. For a chosen combination $n = 10$, $m = 3$, and $N = 3$, it turns out that the proposed algorithm requires 4085 operations against 13 050 operations of the other algorithm [SI]. This demonstrates the computational superiority of the proposed algorithm. It is also of interest, to note that, starting with (5) it is possible to derive an alternate dynamic form of asymptotically stable lag smoother that is also computationally efficient. This will be indicated elsewhere.

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Frequency Criteria for the Absence of Limit Cycles in Nonlinear Systems

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Abstract—A method is derived herein that provides a geometric interpretation of frequency regions in the Nyquist plane over which simple oscillations cannot occur. These criteria answer a question of system behavior when Popov-type criteria are violated.

INTRODUCTION

Although satisfaction of the various frequency domain stability criteria guarantees asymptotic stability of nonlinear systems, quite often one is interested in determining criteria for the nonexistence of periodic solutions. The purpose of this correspondence is to delineate conditions under which the system will not support an assumed limit cycle.

The criterion supplies a tradeoff between the "amount" by which the Popov criterion may be violated and the guarantee that, if a limit cycle exists, it must be at a fundamental frequency lower than a certain value. Consider the system of Fig. 1, where $G(D)$, $D \frac{d}{dt}$, is a linear time-invariant operator and $f(\cdot)$ is differentiable odd, monotonic, and memoryless, i.e., where

$$0 \leq o(t)f[a(t)] \leq K_1 \sigma^2(t) \tag{1}$$

$$0 \leq \frac{d\sigma}{dt} \leq K_2 \tag{2}$$

The main result is stated in the following.

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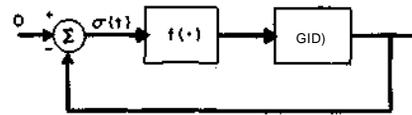


Fig. 1.

Theorem: Suppose there exists a $q \in R$ and a $n \in R^+$ such that

$$\left[X(jn\omega) + \frac{1 + \alpha(n\omega)^* K_2}{2[K_1 + \alpha(n\omega)^2 K_2]} \right] a + \left[Y(jn\omega) + \frac{nq}{2[K_1 + \alpha(n\omega)^2 K_2]} \right]^2 < \frac{[1 + \alpha(jnti)^*] + (n\omega q)^2}{4[K_1 + \alpha(n\omega)^2 K_2]^2}, \tag{3}$$

V odd n

where $X(j\omega) = \text{Re} [G(jnG)]$ and $Y(j\omega) = \text{Im} [G(jnG)]$. Then the system of Fig. 1 will not support a simple symmetric periodic solution of the form

$$\sigma(t) = \text{Re} \left[A e^{j(\omega t + \phi_n)} \right] \tag{4}$$

n odd

The reader will immediately recognize that the region in (3) is the interior of a circle (for fixed G , q , and a) in the X - Y plane.

Proof: Consider Fig. 1.

$$\text{to be } = -G(D)f\sigma(t) \tag{5}$$

From conditions (1) and (2) we have

$$\langle r(t)[K_1 \sigma(t) - f\sigma(t)] + a[K_1 \sigma^2(t) - f\sigma(t)] \geq 0 \tag{6}$$

for $a \geq 0$. Now, subtracting $qu(f)Df[u(t)]$, $q \in R$, from both sides of (6) and using (5) we get

$$\sigma(t)[K_1 + (1 + nD)G^{-1}(D)ldt] + aD^{-1}(b)[K_2 + G^{-1}(D)]D\sigma(t) \geq -qdt)Df[\sigma(t)] \tag{7}$$

Assuming that $\sigma(t)$ is of the form of (4), and integrating both sides of inequality (7) over one period of oscillation, implies

$$\int_0^{2\pi/\omega} \sigma(t)[K_1 + (1 + nD)G^{-1}(D)ldt] + aDu(t)[K_2 + G^{-1}(D)]Du(t) dt \geq 0, \tag{8}$$

since $\sigma(t)$ and $f[u(t)]$ are both assumed periodic, of period $2\pi/\omega$. Using (4) in (8) then yields

$$\text{Re} \sum_{n \text{ odd}} \{ K_1 + (1 + jn\omega q)G^{-1}(jn\omega) + \alpha(n\omega)^2 [K_2 + G^{-1}(jn\omega)] \} A_n^2 \geq 0. \tag{9}$$

Inequality (9) is a necessary condition for the assumed oscillation (4) to exist. Its violation would, therefore, be sufficient to ensure that no limit cycle of fundamental frequency G could be supported by the system. One obvious violation of (9) is the requirement that

$$\text{Re} [K_1 + (1 + jnGq)G^{-1}(jn\omega) + \alpha(n\omega)^2 [K_2 + G^{-1}(jn\omega)]] < 0, \tag{10}$$

V odd $n \in R^+$

Using $G(jnG) = X(jnij) + jYW$ in (10) implies (3), and the proof is complete.

Now referring to inequality (3), it is seen that the theorem requires that points on the Nyquist plot of $G(\omega)$, corresponding to ω , $n = 1, 3, 5, \dots$, must lie inside a circle (corresponding to n) to ensure that the system will not support a limit cycle of fundamental frequency G .