where

$$K(t_k) = \int^{t_k} P(t_k,s|t_{k-1})H'(s)\,ds$$

$$\cdot\{\{''' \quad [''' \; HO)nt,s\backslash t_k\sim x)HXs)dsdt+RO_k)\}^{-1} \quad (5)$$

$$\frac{d}{dt}P(t,s|t_{k-1}) = F(t)P(t,s|t_{k-1}), \qquad \text{for } t_k \geq t \geq s > t_{k-1} \quad (6)$$

$$\frac{d}{dt}P(t,t|t_{k-1}) = F(t)P(t,t|t_{k-1}) + P(t,t|t_{k-1})F'(t) + G(t)G'(t),$$

tor f£ > / > /fc — | (7)

$$P(t,s|t_{k-1}) = P'(s,t|t_{k-1}), \qquad \text{for } t_k>s> t> t_{k-1} \quad (8)$$

and

$$P(t_k,t_k|t_k) = P(t_k,t_k|t_{k-1}) - K(t_k)\int^{t_k}_{t_{k-1}} H(s)P(s,t_k|t_{k-1})\,ds. \quad (9)$$

Here, the prime denotes matrix transpose.

Derivation of the Algorithm: For linear Gaussian system **(1)** and (2), the minimum-variance estimate $\hat{x}(t_k|t_k) \triangleq E\{x(t_k)|Y^k\}$ can be obtained in the form of

$$\hat{x}(t_k|t_k) = \sum^{k} L(t_k,t_l)\nu(t_l) \quad (10)$$

where $L(t_k,t_l)$ $(l = 1,\ldots,k)$ are nxp matrices to be determined in optimal fashion in the sequel and

$$\nu(t_l) \triangleq y(t_l) - E\{y(t_l)|Y^{l-1}\}. \quad (11)$$

Here, $E\{\cdot|\cdot\}$ denotes conditional expectation and $\{\nu(t_l), l=1,2,\cdots\}$ is an independent sequence and is called innovation process [2]. From (10) we can easily obtain

$$L(t_k,t_l) = E\{x(t_k)\nu'(t_l)\}[E\{\nu(t_l)\nu'(t_l)\}]^{-1} \quad (12)$$

where $E\{\cdot\}$ denotes mathematical expectation. From (11) and (2), we have

$$E\{x(t_k)\nu'(t_k)\} = \int^{t_k}_{t_{k-1}} P(t_k,s|t_{k-1})H'(s)\,ds \quad (13)$$

where

$$P(t,s|t_{k-1}) \triangleq E\{\tilde{x}(t|t_{k-1})\tilde{x}'(s|t_{k-1})\} \quad (14)$$

$$\tilde{x}(t|t_{k-1}) \triangleq x(t) - \hat{x}(t|t_{k-1}) = x(t) - E\{x(t)|Y^{k-1}\}. \quad (15)$$

Furthermore, from (11) and (2)

$$E\{\nu(t_k)\nu'(t_k)\} = \int^{t_k}_{t_{k-1}}\int^{t_k}_{t_{k-1}} H(t)P(t,s|t_{k-1})H'(s)\,ds\,dt + R(t_k). \quad (16)$$

From (10) and the property of the innovation process $\nu(t_l)$,

$$\hat{x}(t_k|t_{k-1}) = E\{\hat{x}(t_k|t_k)|Y^{k-1}\} = \sum_{l=1}^{k-1} L(t_k,t_l)\nu(t_l). \quad (17)$$

Therefore, from (10)-(17) the optimal estimate $\hat{x}(t_k|t_k)$ is given by (4) and (5). Also, by (1) and definition (14), matrix $P(t,s|t_{k-1})$ in (5) satisfies (6)-(9). Moreover, from (1), prediction $\hat{x}(t|t_k)$ $(t>t_k)$ is given by (3). This completes the derivation of the algorithm.

IV. Concluding Remarks

We have derived the minimum-variance estimator algorithm for a linear continuous-discrete system with noisy state-integral observations by an innovations approach. The presented optimal estimator has the similar structure to the usual Kalman filter [1, 3]. Difficulty for the practical implementation of the optimal algorithm is that we should carry out the integration in (4), (5), and (9). Therefore, practically we should discretize the time interval of interest in (4), (5), and (9) and then replacing the integration by the summation we have a feasible approximate estimator algorithm.

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Least-Squares State Estimation in Time-Delay Systems with Colored Observation Noise: An Innovations Approach

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Abstract—The method of innovations approach has been applied to develop an algorithm for the least-squares state estimation of a nonstationary linear discrete system with multiple timedelays, based on observations involving multiple time-delays and colored noise.

I. Introduction

The problem of optimal filtering and smoothing of nonstationary linear discrete systems with multiple time-delays has been considered by several authors [1]-[5]. The aim, always, has been to derive Kalman-type equations. They have either used the method of orthogonal projection [1],[2] or have used standard Kalman filter results after augmentation and finally decomposing to component equations [3]-[5].

Priemer and Vacroux have considered the system with white plant and observation noises in [1] and have extended the results to include fixed lag smoothing in [2]. The alternate method of state augmentation has been used by Farooq and Mahalanabis [3] to obtain same results as in [1] and by Biswas and Mahalanabis [4],[5] to obtain algorithms for optimal smoothing with added complexities to the problem.

The aim of this correspondence is to develop Kalman-type equations for a more general case when the observations are both colored and delayed. The technique of innovations approach [6], which is decidedly more convenient and elegant, has been used for this purpose. This technique is a different one than those used so far. It has been pointed out, as a major conclusion of the present study, that all the results [1]-[5] are obtainable from the results derived here under different assumptions.

II. Problem Statement

The processes under consideration are described by the vector difference equations of the form

$$x(k+1) = \sum_{i=0}^{M} \varphi_i(k)x(k-i) + u(k) \quad (1)$$

and the measurement equation is given by

$$y(k) = \sum_{i=0}^{N} H_i(k)x(k-i) + r(k), \qquad k=0,1,2,\cdots \quad (2)$$

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where x is the n-vector state, y is the m-vector output, $\varphi_i(k)$ and $H_i(k)$ are time-varying matrices of order $n \times n$ and $m \times n$, respectively, and M and N are the total number of delays in system and observation, respectively. The m-vector observation noise $r(k)$ is assumed to be colored sequence with the following propagation model:

$$r(k+1) = B(k+1)r(k) + v(k+1). \quad (3)$$

The noise sequences $u(k)$ and $v(k)$ are assumed to be independent, zero mean, white, and Gaussian, so that

$$\overline{u(k)u'(j)} = Q(k)\delta_{kj}$$

$$\overline{v(k)v'(j)} = R(k)\delta_{kj}$$

$$\overline{u(k)v'(j)} = 0, \qquad \text{for all } j, k > 0$$

where the bar denotes expectation, prime denotes transpose, δ_{kj} is the Kronecker delta, Q and R are positive-definite matrices of order $n \times n$ and $M \times m$, respectively.

The initial states are zero-mean Gaussian random vectors which are independent of $u(k)$ and $v(k)$ and

$$\overline{x(-j)x'(-l)} = P_0(j,l),$$

$$j = 0,1,\cdots,J;\ l = 0,1,\cdots,J;\ J = \max\{M,N\}. \quad (4)$$

The problem is to develop Kalman-type algorithm for the optimal estimate of the state $x(k-j)$, $(j=0,1,\cdots,M)$ for given observations up to k, and then extend these results to include optimal smoothing.

III. LINEAR FILTERING

Unlike Kalman approach, the innovations method does not directly deal with the observation process $y(k)$, but instead with an equivalent innovations process $\nu(k)$ conveying the same statistical information.

$$\nu(k) \triangleq y^1(k) - \hat{z}(k/k-1) = \tilde{z}(k/k-1) + v(k) \quad (5)$$

where

$$y^1(k) \triangleq y(k) - B(k)y(k-1)$$

$$z(k) \triangleq \sum_{i=0}^{N} H_i(k)x(k-i) - B(k)\sum_{l=0}^{N} H_l(k-1)x(k-l-1)$$

and $\hat{z}(k/k-1)$ is the linear least-squares estimate of signal $z(k)$ given $(y(l),\ 0 \le l \le k-1)$ and $\tilde{z}(k/k-1)$ is the corresponding estimation error. Therefore,

$$\nu(k) = \sum_{i=0}^{N} H_i(k)\tilde{x}(k-i/k-1)$$

$$- B(k)\sum_{l=0}^{N} H_l(k-1)\tilde{x}(k-l-1/k-1) + v(k). \quad (6)$$

It has been proved in [6] that

$$\overline{\nu(k)} = 0, \qquad \overline{\nu(k)\nu'(l)} = [P_z(k) + R(k)]\delta_{kl}$$

where the covariance of estimation error is given by

$$P_z(k) = \sum_{i=0}^{N}\sum_{l=0}^{N} H_i(k)P(k-i,k-l/k-1)H_l'(k) - U(k) - U'(k)$$

$$+ B(k)\sum_{i=0}^{N}\sum_{l=0}^{N} H_i(k-1)P(k-i-1,k-l-1/k-1)H_l'(k-1)B'(k) \quad (7)$$

where

$$U(k) \triangleq \sum_{i=0}^{N}\sum_{l=0}^{N} H_i(k)P(k-i,k-l-1/k-1)H_l'(k-1)B'(k)$$

and

$$P(i,l/k) \triangleq \overline{\tilde{x}(i/k)\tilde{x}'(l/k)}. \quad (8)$$

By utilizing the projection theorem it is trivial to prove that

$$\hat{x}(k/b) = \sum_{l=0}^{b} \overline{x(k)\nu'(l)}\,[P_z(l) + R(l)]^{-1}\nu(l), \qquad \forall b. \quad (9)$$

Using (9) and (1), it can be proved that

$$\hat{x}(k+1/k) = \sum_{i=0}^{M} \varphi_i(k)\hat{x}(k-i/k). \quad (10)$$

Substituting $k-j$ for k and k for b in (9) and taking summation from 0 to $k-1$ and adding to it the kth term

$$\hat{x}(k-j/k) = \sum_{l=0}^{k-1} \overline{x(k-j)\nu'(l)}\,[P_z(l) + R(l)]^{-1}\nu(l)$$

$$+ \overline{x(k-j)\nu'(k)}\,[P_z(k) + R(k)]^{-1}\nu(k).$$

The first term equals $\hat{x}(k-j/k-1)$ from (9), therefore,

$$\hat{x}(k-j/k) = \hat{x}(k-j/k-1) + K_j(k)\Big[y^1(k) - \sum_{i=0}^{N} H_i(k)\hat{x}(k-i/k-1)$$

$$- B(k)\Big\{y(k-1) - \sum_{l=0}^{N} H_l(k-1)\hat{x}(k-l-1/k-1)\Big\}\Big],$$

$$j = 0,1,\cdots,J \quad (11)$$

where

$$K_j(k) \triangleq \overline{x(k-j)\nu'(k)}\,[P_z(k) + R(k)]^{-1}.$$

Expressing $x(k-j)$ as sum of $\hat{x}(k-j/k-1)$ and $\tilde{x}(k-j/k-1)$ and substituting the value of $\nu(k)$ from (6),

$$K_j(k) = \Big[\sum_{i=0}^{N} P(k-j,k-i/k-1)H_i'(k)$$

$$- \sum_{l=0}^{N} P(k-j,k-l-1/k-1)H_l'(k-1)B'(k)\Big]$$

$$\cdot [P_z(k) + R(k)]^{-1}, \qquad j = 0,1,\cdots,J. \quad (12)$$

From (1) and (10),

$$\tilde{x}(k+1/k) = \sum_{i=0}^{M} \varphi_i(k)\tilde{x}(k-i/k) + u(k).$$

Therefore,

$$P(k+1,k+1/k) = \sum_{i=0}^{M} \varphi_i(k)P(k-i,k+1/k) + Q(k) \quad (13)$$

and

$$P(k-j,k+1/k) = \sum_{i=0}^{M} P(k-j,k-i/k)\varphi_i'(k), \qquad j = 0,1,\cdots,J. \quad (14)$$

utilizing (Il),

$$P(k-j,k-i/k)$$

$$= \tilde{[}.;((k-j/k- 1) - K,(k)v(k)\tilde{]}[\sim (k-i/k-l)- K,(k)_v(k)l,$$

Substituting the value of $K,(k)$ from (12) and $v(k)$ from (6).

$$P(k-j,k-i/k)=P(k-j,k-ijk-l)-\sim\sim;(k)$$

$$\cdot\left[\sum_{l=O}^{N} H,(k)P(k-I,k-i/k--l)\right.$$

$$\left.-\sum_{1=0}^{N} B(k)H,(k-\backslash)P(k-l-l,k-i/k-l)\backslash,\right]$$

$$7=0,1,\cdots,./;\quad i=O;l,--J. \quad (15)$$

Equations (l0) 415) form the required recursive chain starting n-ith the initial conditions $\hat{i}(-j/-1)=0$ a s y (- 1) is nonexistent; hence,

$$P(-i,-j/-1)=P_0('V). \quad ./=0,1,\cdots,/; \quad '=0,1,\cdots j. \quad (16)$$

I\'. O P R W SMOOIHING

Due to time-delay involved, the problem of smoothing is mherently present in the problem of filtering. Specifically, the smoothing problem is the determination of $\hat{i}(k- L/k)$ where $/<L<X$. To utilize the results of optimal filtering for smoothmg, the system equation (I) is considered as

$$x(k+1)=\sum^{L} \varphi_i(k)x(k-i)+u(k) \quad (17)$$

where

$$\varphi_i(k)\equiv 0, \qquad \text{for } i=M+1,\cdots,L.$$

Now for (17) the recursive algorithm can directly be applied, with the given initial conhtions, in three stages. First, the iterations continue as in the case of linear filtering till $k=J$ with $0=O, I;..J;$ $i=O, l;..\cdot j)$. Second, for further iterations, the limit of j is increased by unity in each cycle til1 $k=L$. The last iteration with $k=L$ gives the *fXxed-infen;a/ (L)-smoorhed* values of state. And third, the iterations continue without any more increase in the limit of j. For the last two stages, $j=0,l.\cdot..,J;$ $i=0,1,-\cdot\cdot J$ and for $7=/+I,-\cdot\cdot,q;$ $i''=0,1,-\cdot\cdot,/;$ where $q=min(k,L)$. 1c is obvious that these iterations will give *fixed-/ug-sntooihed* values.

However, if the filtering algorithm is continued up to $k=x+/$ and from this point onwards the index limit is increased so as to include point s only $0=O, I;..,J.k-s;$ $i=O,1,\cdot.j)$, the result is *fiwed-point snroozhing* at s.

V. CONCLUSION

It has been found that the optimal filtering algorithm is very conveniently derived using the innovations approach. The results are also applicable *to* all three classes of smoothing by a slight modification in the range of the indexing variable. Results obtained earher in [IH5] are directly obtained from (10)115) by assorning B and/or h' equal to zero as the case may be.

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Rapid Estimation and Detection Scheme €or Unknown Discretized Rectangular Inputs

R. V. SONALKAR AND C. N. SHEN

***Abstmct*—An algorithm to deteci unknown discretized rectan&u inputs and to estimate the state simultaneously, has been descn'bed in this note. Tbe discretized redangnlar input is represented by a sequence of equal mgnitnde inputs to the state. Four alternate hypotheses are formed at every stage of observation and Bayes risks are compared to determine the corpect one. A minimumi variance estimate of the inputs is used to improve the estimate of the state. Computer results from a numerical example are shown to demonstrate that a possible divergence of the Kalman filter can be prevented by incorporating the detection scheme.**

I. INTRODUCTION AM) PROBLEM STATEMENT

In [I], an algorithm to detect a single isolated input and obtain better estimates of the state by making use of the estimated value of the input, is discussed. Reference [2] and this note are extensions of [l] and demonstrate the detection of a train of inputs. The assumption involved in this note is that at least three consecutive equal magnitude inputs occur. Also, more than one input sequence can be detected if consecutive sequences are separated by at least three observation instants. The system dynamics are given by the following equations:

$$x_{i+1}=F_ix_i+w_i+\sum_{j=k}^{k+r} Bu_j\delta_{ij} \quad (1)$$

$$z_i=H_ix_i+v_i \quad (2)$$

*here x_i and $wi€R''; ujER^q$ is the unknown but deterministic input vector; z, and c, E R^S. Both k and r are unknowns and must be detected. The initial state x_o, the plant noise w_i, and the observation noise u_t are assumed to be Gaussian with statistics $\hat{N}(i_o,P_o)$, $N(O,Qi6J.$ and $h^7(Oo,Rz6,.)$, respectively, and they are mutually uncorrelated. It is also assumed that *sa q* and that the matrices $_4$, B, H_i, Q_i, and R, are known *a priori.* The matrices Hi and B are of full rank, R, is positive definite, and Qi positive semidefinite.

The input train is said to have started at stage i if the first input occurs between stages i and $i+1$. At every stage of observation, four different hypotheses are formed which are as follows [2]:

h,: Input train started at stage $i+m-1$, denoted by branch M in Fig. 1, $m=1$, 2, 3.

h,: Impulse train started at stage i + 3 or later, branch a, Fig. 1.

Branch *6* in tree I1 is a continuation of branch 1 in tree I for the special case of the sequence starting at i, assuming that the sequence of inputs continues. The other three branches *a*, *b*, and c represent three hypotheses, h_a, h_b, and h, similar to *h,, h,,* and *h,,* and refer to the

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