Delay Time Sensitivity in Nonlinear Monotone *RC* Trees

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Abstract—Sensitivity of delay time, *T* (time required to achieve a given target voltage) at any node of a nonlinear monotone *RC* Tree is studied using the adjoint network approach. It is shown that the sensitivity of *T* with respect to changes in a parameter of the resistors or the capacitors of the tree can be expressed as an integral of the solutions of the network and its adjoint network. Using this integral it is then established that r increases for some nodes (depending upon location of a node) due to the decrease in a given resistor value, when T is small. For other nodes T would decrease. However, if the target voltage is close to the steady-state value then *r* decreases for all the nodes.

I. INTRODUCTION

In a digital MOS integrated circuit, a given inverter or a logic node may drive several gates, some of them through long wires having distributed resistances, capacitances, and pass transistors [1]-[3]. Signal delay at the input of various gates are computed by a timing simulator. Modeling a digital gate and its fan-out interconnection including pass transistors as an *RC* network (a resistive network with capacitors between nodes and ground) is a well-accepted practice for various studies on the gates [1]-[7]. These networks may be *RC* lines/trees/meshes having linear or nonlinear element characteristics. Signal delay and delay time sensitivity of voltage at any node are important in timing simulation/analysis. Wyatt [8]-[10] studied the problem of sensitivity of the voltage response at any node on an *RC* mesh containing monotone resistor and capacitor elements. He proved that sensitivity of the voltage response at any node on an *RC* mesh containing monotone resistor and capacitor elements is monotonic with respect to the overestimation or the underestimation of the source resistance, grounded resistances, and any capacitance [8]. For a *RC* line [9] he showed that the sensitivity of the output node voltage of the line is also monotonic for any resistance on the line. These are only partial results. For example, we do not know whether similar results hold for a tree. Similarly, we do not know what will happen to the sensitivity if a resistor other than the source resistance changes. In this paper we provide answers to these questions.

In a monotone *RC* tree we show that for a monotone resistor R_{lk}, between the nodes *j* and *k*, if the magnitude of current through the resistor R_{jk} increases by increasing one of its parameters (say p_{jk}), then the delay time $T_{(}$, at any node *e* decreases for an increase in p_{jk} for the nodes *j* and *k* located on the path between the source and the node *e* and increases for all other nodes, if a small 7„ is under consideration. For large *T*,, (the target voltage is close to the steady-state value), the sensitivity of 7„ is negative. This is true for all nodes. (This problem for a linear *RC* tree is studied in [11].)

II. NOTATIONS AND ASSUMPTIONS

We first introduce some notations [8], Let *N* be an *RC* mesh network consisting of a finite number of two-terminal resistors and two-terminal capacitors driven by a single source, connected in such a way that a single node (ground) terminates one side of every capacitor and one side of the source. The voltage source is assumed to be between the node $(n+1)$ and 0 (ground).

Let $i_{lk}($ r, — i'_k. $p_{ik})$ denote the current through the resistor branch (R_{jk}) from the nodes *j* to *k* having a voltage difference of (/', - *i*\), where p_{jk} denotes one of the parameters of interest (such as area) of (R_{jk}) such that

$$p_{jk} = p_{kj} \qquad 0 \leq j, k \leq n+1. \qquad (2.1)$$

Since $i_{jk}(\cdot)$ and $i_{kj}(\cdot)$ describe the same resistor branch, we have for any value of the voltage *v* across R_{jk}:

$$',*(''. Pjk) = -'*/(''''\cdot Pkj)- \qquad (2-2)$$

For a monotone R_{lk} it is assumed that: i) $i_{lk}(\cdot)$ is continuously differentiable with respect to *v* as well as p_{lk}; ii) it passes through the origin; and iii) is a nondecreasing function of *v*, i.e.:

$$',*(0, pjk) = 0$$

and

$$[di_{lk}(\cdot)/dv] \geq 0. \qquad \textbf{(2.3)}$$

Let *Cid'j, p:)* be the capacitance between the nodes *j* and the ground. Here *Pj* denotes one of the parameters of interest of C_r For a monotone capacitor, it is assumed that the capacitor characteristic is increasing function, i.e.:

$$<:,(\cdot) > 0; \qquad 1 \leq j \leq n. \qquad \textbf{(2.4)}$$

It is also assumed that all the parameters of resistors and capacitors are independent.

Further, for uniqueness of the solutions of an *RC* mesh containing monotone resistors and capacitors, it is required that $i_{jk}(\cdot)$, $[di_{jk}(\cdot)/dv]$, $[1/C,(\cdot)1$, $[d[l/Cj(\cdot)]/dr]$ satisfy a global Lipschitz condition.

The state equations of an *RC* mesh *N* can be written as

$$q(v_r\ p_i)\dot{v}_j + \sum_{k \in A(j)} i_{jk}(v_j - v_k, p_{tk}) = 0 \qquad \textbf{(2.5)}$$

where

$$j = 1.2, \cdots, ,,;r,(0) = r_{/0}$$

$$* = 0, 1, 2, \cdots, n, (n+1)$$

and

$i'o(t) = 0$; for the ground node,

$t'_{n+1}(t) = u(t)\backslash$ the source voltage,

A (j) denotes the set of nodes adjacent to the node *j*.

It is assumed that *u(t):* i) is continuously differentiable, ii) is a nondecreasing function, and iii) approaches a constant *a t* tends to infinity.

It is assumed that *u(t)* is continuously differentiable and approaches *u{* 00).

The following results given by Wyatt [8] are required to prove the main result of our paper. Assume that all the capacitors in *N* are in the charging mode and the excitation *u(t)* is nondecreasing.

1) Monotonicity in time [8|: the voltage response at any node is nondecreasing and bounded by *u(t)* in a monotone *RC* mesh.

2) Spatial monotonicity [10]: in a monotone *RC* tree:

$$u(t) \geq v_j(t) \geq v_k(t) \qquad \textbf{(2.6)}$$

where

$$j = P(k)$$

p(k) (called the parent node of *k)* denotes the adjacent node towards the source node from the nodey.

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Define an event functional [12], [13|:

$$\mathrm{EV}\ [z(t)] = \begin{cases} 1, & z(r) < 0 \\ 0, & z(r) \geq 0. \end{cases} \quad \textbf{(2.7)}$$

The event is recognized whenever the voltage $i\backslash.(t)$ of the node e attains the target voltage K_{Tllr}. Thus the time delay for any node e of interest is

$$\tau_{e} \doteq \int_0^T \mathrm{EV}\ [v_e(t) - V_{Jm}\backslash dt \quad (2.8)$$

where T is very large with respect to r,..

Using (2.8) and the monotonicity of the voltage response, the delay time r_c can be qualified as "small" T,. or "large" T„ with respect to a target voltage (K_{Tar}) in the following sense.

1) Small r_e: When the target voltage K_{Tar} for a node e is close to the initial condition (',,(0), the delay time (T,,) required to achieve the voltage level V_{Tiir} would be small.

2) Large T,.: When the target voltage K_{Tar} for a node e is close to the final voltage «(o°), the delay time (T,,) required to achieve the voltage level K_{Tar} would be large.

III. Delay Time Sensitivity

Consider a circuit described by the following system of differential and algebraic equations [12]:

$$f(x, \dot{x}, y, p, I) = 0. \quad (3.1)$$

In this equation, x is a vector denoting the variables (i.e., currents and voltages) whose first derivatives with respect to time appear in the circuit equilibrium equations (in general these are the state variables), $\dot{x}$ is the vector of time derivative of x, y is a vector which includes other relevant current and voltages not included within x, and/> is a vector of design parameters, i.e., those parameters which are to be regarded as subject to change, and t is time.

Consider a functional of circuit variables:

$$\Phi(p) = \int_0^T 0(x,\dot{x},y,p, t)dt. \quad \textbf{(3.2)}$$

The initial condition on x and the final time T are assumed to be independent of the design parameters $p\backslash$ i.e., $[dx(0)/dp] = [dT/dp] = 0$. Differentiation of (3.2) and adoption of new notations yields (for a detailed derivation see [12])

$$\frac{d\Phi}{dp} = \%(p) = \int_0^T (0_p + f_p' \lambda)dt \quad (3.3)$$

where X|(?) and X_2(O will satisfy the following equations:

$$\dot{x}_2 = f_x^{\prime} x_1 + 0_x,$$
$$0 = f_s' x_1 - x_2 + \varnothing_s$$
$$0 = f_y^{7} x_1 - 0_y \quad \textbf{(3.4)}$$

with

$$\lambda_2(T) = 0 \quad \text{and} \quad 0 \leq t \leq T.$$

and

$$s(t) \doteq \dot{x}(t)$$

$$f_x \doteq [df(\cdot)/dx], \qquad f_s = [df(\cdot)/ds]$$

$$f_y \doteq [a/(\cdot)/a.v], \qquad f_p \doteq [df(\cdot)/d_P]$$

$$0_x \doteq [d0(\cdot)/dx], \qquad 0_S \doteq [d0(\cdot)/ds].$$

$$0_y \doteq \backslash d0(\cdot)/dy].$$

Equation (3.4) is a system of differential and algebraic equations called the adjoint equations. They are linear equations with forcing terms given by the derivatives of (0), the integrand of the function whose sensitivity is to be computed. In order to obtain X, as a function of time (for substitution into (3.3) to find $(d\Phi/dp)$), the set of equations (3.4) must be solved. Since the initial conditions on $\lambda_2(T)$ are given at time 7', (3.4) are solved starting from $t = T$ down to $t = 0$; i.e., the solution proceeds backward in time for $T \geq t \geq 0$. Note that the adjoint system (3.4) are linear but time varying. The matrices f_v, f_s and f, are time varying, since in general they depend on $x(t)$, $s(t)$ and $y(t)$, the solutions of the circuit equation (3.1).

For an /?Cmesh described by the system of differential equations of (2.5), x is the node voltage vector; i.e.:

$$x = V = [v_1, v_2 \cdots v_n]^T \quad \text{and} \quad y = 0.$$

The integrand (0) for the delay time T_e can be obtained by comparing (2.8) and (3.2). Thus

$$\varnothing = \mathrm{EV}\ [v_e(t) - V_{Tar}]. \quad \textbf{(3.5)}$$

Therefore,

$$0_s = 0, = 0$$

and [12], [13|:

$$0_x = -\delta(t - T_e)[1/\dot{v}_e(T_e)]e_r = -\delta(t - T_e)[1/\dot{v}_e(T_e)]e_r \quad (3.6)$$

where $h(t - T_e)$ is the delta function at $t = T_e$, and

$$e_e = [0 \cdots 010 \cdots 0]'.$$

Since $0_x = 0$ for $T_e < t \leq T$, the initial condition $\lambda_2(T) = 0$ is equivalent to $\lambda_2(T_e) = 0$.

Further from (2.5) for an RC mesh:

$$f_x = E(t). \quad f_s = C(t), \quad \text{and} \quad f_y = 0 \quad (3.7)$$

where $E(t)$ and $C(1)$ are

$$C(t) = \mathrm{diag}\,[C_1(v_1, p_1) \quad C_2(v_2, p_2) \cdots C_n(v_n, p_n)] \quad (3.8)$$

and (j, w)th entry of the matrix $E(t)$ is

$$E_{jw}(t) = \begin{cases} \dfrac{\partial C_j}{\partial v_j}\dot{v}_j + \sum_{k \in A(j)} \dfrac{\partial i_{jk}}{\partial v_j}, & j = w \\ -\dfrac{\partial i_{jk}}{\partial v_k}, & \text{if } = k \in A(j); \ k \neq 0.\ (n+1) \\ 0, & w \notin A(j).\ iv \neq k \end{cases}$$
$$j, w = 1, 2, \cdots, n. \quad \textbf{(3.9.1)}$$

Let the parameter /; be a scalar. For an RC mesh p can be one of following: it can be a parameter:

i) Pj of a capacitor C_j:
ii) p,o of a grounded resistor R_{j0};
iii) $p_{j,n+1}$ of a source resistor $R_{j,n+1}$;
iv) Pik of any resistor R_{jk}

where $1 \leq j, k \leq n$.

Therefore, from (3.5)

$$0_p = 0 \quad \textbf{(3.9.2)}$$

and from (2.5) for any node $j = 1, 2, \cdots, n$:

$$f_p = \begin{cases} [\partial C_j(\cdot)/\partial p_j]\dot{v}_j(t)e_j, & p = p_j \\ [di_{j0}(\cdot)/dp_{j0}]e_r & p = p_{j0} \\ -[\partial i_{j,n+1}(\cdot)/\partial p_{j,n+1}]e_j, & p = p_{j,n+1} \\ [\partial i_{jk}(\cdot)/\partial p_{jk}][e_j - e_k], & p = p_{jk} \end{cases} \quad (3.9.3)$$
$$k = 1, 2, \cdots, n$$

where

$e_j = [00 \cdots 010 \cdots 0]'$ and $e_k = [00 \cdots 010 \cdots 0]'$.

Substituting (3.9.2) and (3.9.3) into (3.3), the sensitivities of the delay time r,, [12] with respect to the parameter of any one of the capacitors or resistors are obtained as

$$\frac{dr_c}{dp_j} = \int_0^{T_c} \frac{dC_j(\cdot)}{dp_j} \dot{v}_j \lambda_{1j}(t)\, dt \qquad (3.10.1)$$

$$\frac{\partial \tau_c}{dPjo} = \int_0^{\tau_c} \frac{di_{in}(\cdot)}{\partial p_{j0}} \lambda_{1j}(t)\, dt \qquad (3.10.2)$$

$$\frac{dr_c}{} = -\int_0^{\tau_c} \frac{\partial i_{(n+1)j}(\cdot)}{\partial p_{(n+1)j}} \lambda_{1j}(t)\, dt \qquad (3.10.3)$$

$$\frac{9\wedge}{\partial p_{jk}} = \int_0^{\tau_c} \frac{\partial i_{jk}(\cdot)}{\partial p_{jk}} \left[\lambda_{1j}(t) - \lambda_{1k}(t)\right] dt \qquad (3.10.4)$$

where $1 \le j, k \le n$:

$$\dot{\lambda}_2(t) = E(t)\lambda_1(t) - \left[1/\dot{v}_c(\tau_c)\right]\delta(t-\tau_c)e_c$$

$$0 = C(t)\lambda_1(t) - X_2(r)$$

with

$$X\_,(T,,) = 0; \quad 0 \le ; \le T,, \qquad (3.11)$$

is the adjoint system. Note that the initial condition is $\lambda_2(T_c) = 0$. Therefore, the upper limit of integration in (3.10.1)-(3.10.4) is r,..

Further:

$$\dot{\lambda}_2(t) = \left[\frac{\partial C(\cdot)}{dp}\frac{\partial p(t)}{dt} + \frac{d}{dV}\wedge\dot{V}\right] X_1(0 + C(t)\dot{\lambda}_1(t).$$

It is assumed that all the parameters are time invariant. Thus

$$\dot{M'}) = \frac{3C(\cdot)}{dV}\dot{V}\lambda_1(t) + C(t)\dot{\lambda}_1(t). \qquad (3.12)$$

Substituting (3.12) into (3.11)

$$\left[E(t) - \frac{dC(\cdot)}{dV}\dot{V}\right]\lambda_1(t) - \left[1/\dot{v}_c(\tau_c)\right]\delta(t-\tau_c)e_c$$

with

$$X,(T,,) = 0 \text{ (since } C(f) * 0) \text{ and } 0 \le / \le T,. \qquad (3.13)$$

The adjoint system described in (3.13) is a linear time varying system having $[1/\dot{v}_c(T_c)]\delta(t)e_e$ as the forcing function. $(1/\dot{v}_c(T_c))\delta(t)$ is an impulse function with a strength of $[1/\dot{v}_c(T_c)]$.

Let $X_1^{nr}(t)$ denote the response of the adjoint system having a normalized forcing function. Then

$$K(') = \frac{M')}{\left[1/\dot{v}_c(\tau_c)\right]}. \qquad (3.14)$$

(Note that for a finite T,., i.e., $K_{T_c r} < \infty$, /',.(r,.) ^= 0):

$$C(t)\dot{\lambda}_1^{nr}(t) = G(t)\lambda_1^{nr}(t) - \delta(t-\tau_c)e_c \qquad (3.15)$$

with

$$X',''(T,,) = 0. \quad 0 \le t \le T,$$

and (,/'. ɯ)th entry \>f the $G(t)$ matrix is obtained from (3.9.1) as

$$G_{jw}(t) = \begin{cases} \sum\limits_{k \in A(j)} \dfrac{\partial i_{jk}(t)}{\partial v_j}, & j = w \\ \dfrac{\partial i_{jk}(t)}{\partial v_k}, & w = k \in A(j),\ k \ne 0,\ (n+1) \\ 0, & w \notin A(j),\ w \ne k \end{cases} \qquad (3.16)$$

$j, w = 1, 2, \cdots, n$.

For a resistor element R_{jk} in the RC mesh N, let

$$''id') = [''/(') - ''«(')] \qquad (3.17)$$

$$g_{jk}(t) = \frac{\partial i_{jk}(v_{jk}(t))}{\partial v_{jk}(t)}. \qquad (3.18)$$

From (2.2). (2.3), and (3.18):

$$*/*(') = Ky(0 > 0. \quad t \ge 0. \qquad (3.19)$$

Using (3.18) and (3.19):

$$\left[\partial i_{jk}/\partial v_k\right] = -g_{jk}(t) = \left[\partial i_{kj}/\partial v_j\right] = -g_{kj}(t) \le 0$$

and

$$\left[\partial i_{jk}/\partial rj\right] = suit) > 0. \quad r \ge 0. \qquad (3.20)$$

By substituting (3.20) into (3.16). it can be seen that $G(t)$ is a symmetric time varying conductance matrix having positive conductances.

As the solution of the adjoint system is evaluated backward in time, we introduce new variables d and $X(d)$ as follows:

$$X(\theta) = X_1^{nr}(\tau_c - \theta) \qquad (3.21)$$

$$\theta = \tau_c - t \qquad (3.22)$$

and $\dot{X}(\theta) = dX(\theta)/d\theta$ and $0 \le \theta \le \tau_c$. It is evident from this that $\theta = \tau_c$ corresponds to $t = 0$ and $\theta = 0$ corresponds to $t = T_c$.

Substituting (3.21) into (3.15), we have

$$C(T_c - \theta)\dot{X}(\theta) + G(T_c - \theta)X(\theta) = \delta(\theta)e_c. \qquad (3.23)$$

with

$$X(0) = 0. \quad \text{and} \quad 0 \le \theta \le T_c.$$

The adjoint system represented by (3.23) is an RC mesh N_t with unit impulse current at the node e and $X(d)$ is its node voltage vector. The topology of N_t is same as that of the /fCmesh N, where each nonlinear monotone resistor R_{jt} of N is replaced by a positive linear time varying conductance as given by (3.16) and each nonlinear monotone capacitor of/V is replaced by a positive linear time varying capacitor as given in (3.8). The voltage source $u(t)$ in N is replaced by ground in N_w.

IV. RC MESH

Using the formulations in (3.23). the following Theorem can be proved for the adjoint network N_H.

Theorem I: $X_1^{nr}(t)$ is non-negative: $t \ge 0$ and $X_1^{nr}(0)$ is infinitesimally small as $\tau_c \to \infty$.

Proof: The adjoint network N_H is

$$\dot{X}(\theta) = B(\theta)X(\theta). \quad 0 \le \theta \le T, \qquad (4.1)$$

where

$$X(0) = [1/\{C,(T,)\}]C_{(}. \quad (4.2)$$

$$B(0) = -C'V,. - 0)G(T, - 0). \quad (4.3)$$

Bid) is real symmetric continuous matrix. Since B,-;(0) $\geq$ 0; / =£ *j* and X(0) $\geq$ 0, from the non-negativity theorem of |14], it follows that X(0) $\geq$ 0; 0 $\geq$ 0.

C,(T_r - 0) > 0; *j* = 1, 2, • • • . *n* and G(T,, - 0) is positive definite, the eigenvalues of *Bid)* are real, distinct, and negative [15]. Furthermore for any node/ = 0. 1. • • • , ;i and A- = 0. 1.2, • • - , (« + 1):

$$r_{mih} \leq r_{kj}(t) < M(OO) \quad (4.4)$$

i.e. the region of operation on the characteristic curves for both the resistors and capacitors is bounded. Let the slopes of the *i* - *r* and *q* — *r* curves of all the resistors and capacitors be between a_{lmn}. a_{max}, and $7r_{mm}$, $7r_{lux}$. respectively, thus

$$7T_{mln} \leq C,(/) \leq T_{nm} \quad (4.5)$$

$$o_{mm} £ !i_{jk}(t) \leq a_{miix}. \quad (4.6)$$

It follows that there exists a constant *M* > 0 such that all the eigenvalues of *Bid)* are <— M. Hence, by the stability theorem of [16]:

$$||X(0)|| < \exp[-M0]X(O) \leq \exp[-M0][1/TT,,,,,,] \quad (4.7)$$

i.e. the solutions of (23) are stable and infinitesimally small as 0 → oo.

Using (3.10. l)-(3.10.3). Theorem 1, and monotonicity in time [8], we have the following corollary. This is originally due to Wyatt 19].

Corollary 1: In a monotone *RC* mesh with nondecreasing excitation *u(t),* initially charging capacitors and assuming:

i) *r* • *]di_{jn}(i', Pi_n)/dp_{in}]* $\geq$ 0, for grounded resistors;
ii) *v* • [d/(/, + i),•(«'./>(„ n , /) / d / w u/1 s 0, for the source resistor;
iii) *[dC_t/dpj]* S: 0, for the capacitor at any node *j*

the sensitivity of the delay time (T,.) is non-negative, i.e.:

$$[dr,./dp_{l()}], \quad [drjdp,,,, \ ,,], \quad [drjdp,] \geq 0. \quad (4.8)$$

V. *RC* TREE

Consider a node *e* of interest of an *RC* tree, where the sensitivity of delay time ($T_($,) is to be evaluated with respect to the parameter Pj_k of a resistor element R_{jk}. Let *j* = *pik).* The node *k* falls into the following categories: (a) *k* = *p'"(e),* where *in* is some nonnegative integer; i.e., when the resistor element R_{jk} is located on the path between the source node and node *e.* (b) *k* ^ *p'"(e),* for all values of *m* = 0, 1 ,-••,/;; i.e., when the resistor element R_{ik} is not located on the path between the source node and node *e.*

Theorem 2: The response X',"^/) of the adjoint network of a monotone *RC* tree with unit impulse current at any node *e* has the following properties:

a) small *T,:.*

$$\lambda^{m}_{p(e)}(t) \geq \cdots \geq \lambda^{m}_{pq(e)}(t) \geq 0.$$

and

$$\backslash'i'i(l) \geq \backslash'l'_k(t) > 0; \qquad k \pm p'''(e), \ m = 0, 1, \cdots q.$$

b) large T,.:

$$\backslash TJ(I) < K_kU) > 0 \qquad j = 1,2, \cdots ,n$$

where the node *e* is at the </th position from the source node and *p'"(e): in* = 0, 1. 2, • • • . *q.* are the nodes on the path from the source towards the node *e.*

Proof: For the adjoint network *N,,* let:

H_{ptuui} current from the node *p(a)* towards the node *a;*
H,, current into $C_u(r_t$, — 0).

Since *XJ0)* = 0; *a* * *e.* and X,,(0) = |1/C,,(r,.)] at 0 = 0^+ A¦)(•)• < 0 and n_{eh} > 0 such that *e* = *p(b).* Furthermore:

$$Hi, = V\text{-}ci, \sim \ \underset{h - into}{\wedge} \ Hi,$$

where $\backslash i_{eh}$ ^ $L_{h=,,,,,,}$ $n_{,,h}$ for otherwise *X,,* = 0, while $/t_w$, > 0 implies X_h > X_h (and no X„ is negative, Theorem 1) thus fi_t, > 0 and n_hi, > 0. Using similar arguments, we can show that all the capacitors on the either side of the node *e* in N_0 are in the charging mode at 0 = 0^+. This proves Theorem 2(a). The adjoint network N_e has only one grounded conductance at the node *n* where the source resistor is located in the *RC* tree network *N* between the nodes *(n* + 1) and *n.* Eventually all the capacitors will discharge through this conductance. Thus

$$X,,(0) \geq X_{/)1u)}(0) \qquad \text{large } 0. \qquad a = 1.\ 2, \cdots, n.$$

and Theorem 2(b) follows from it.

We will now use these theorems to prove the following theorem which is the main result of this paper.

Theorem 3: In a monotone *RC* tree with nondecreasing excitation *u(t)* and initially charging capacitors, sensitivity of the delay time T,,, to attain the target voltage K_{Tar} < *u(t)* at any node *e* has the following properties assuming:

$$v \bullet [di_{lk}(i',p_{jk})/dp_{lk}] \ £ \ 0. \quad (5.1)$$

(This implies that an increase in p_{jk} causes an increase in the magnitude of the current through R_{jk}):

$$\frac{\partial \tau_e}{\partial p_{jk}} \begin{cases} £ \ 0, & k = p'''(e) \\ \geq 0, & \text{small } T_(\text{ and } k \ 1= p'''(e) \\ \leq 0, & \text{large } T_, \text{ and for all } k \end{cases} \quad (5.2)$$

where*j* = *p(k), k* = 1, 2, • • • , *n, m* = 0, 1. •• · , <?, and the node *e* is at the *q* position from the source node.

Proof: Let / = *p(k).* Then from the spatial monotonicity theorem of monotone *RC* tree **[10],** we have $i'_{lk}(t)$ > 0. Therefore:

$$\frac{\partial i_{jk}(v_{jk}(t))}{\partial p_{jk}} \geq 0, \quad \text{for } i \geq 0. \quad (5.3)$$

Let

$$\frac{dr..}{\partial p_{jk}}\bigg|^{nr} = [iV(\tau_e)] \frac{\partial \tau_e}{\partial p_{jk}}. \quad (5.4)$$

From (3.10.4), (3.14), and (5.4)

$$\frac{\partial \tau_e}{\partial p_{jk}}\bigg|^{nr} = \int_0^{\tau_e} \frac{\partial i_{jk}(\cdot)}{\partial p_{jk}} [\lambda^{nr}_{lj}(t) - Kk(t)]dt. \quad (5.5)$$

There are two cases for the delay time T, :

Case (i): T„ is small.
From Theorem 2,0 $\leq t \leq r_($:

$$[\lambda^{nr}_{l}(t) - \lambda^{nr}_{lj}(t)] \begin{cases} \leq 0, & k = p'''(e) \\ \geq 0, & k \pm p'''(e). \end{cases} \quad (5.6)$$

Substituting (5.3) and (5.6) into (5.5), we have

$$\frac{\partial \tau_e}{\partial p_{jk}}\Big|^{m'} \begin{cases} \le 0, & k = p^m(e) \\ \ge 0, & k \ne p^m(e). \end{cases} \quad (5.7)$$

Since ṙ,,(T(.) > 0:

$$\frac{\partial \tau_e}{\partial p_{jk}} \begin{cases} \le 0. & k = p^m(e) \\ \ge 0. & k * p^m(e). \end{cases} \quad (5.8)$$

Casf *(ii):* T_L, is large.

The proof for large $T_{(}$, is based on the following facts.

a) In a nonlinear monotone *RC* tree, as f -» o°, each capacitor charges to M(°°) and the voltage across each of the resistors tend to zero. This allows us to construct a linear time invariant limit adjoint *RC* tree from the linear time variant adjoint *RC* tree as shown later.

b) In the adjoint network (which is a linear time variant *RC* tree, with a unit impulse current source at the node *e* and the node *(n +* 1) is at the ground potential), the total charge which flows through a conductance between the nodes *j* and *k* towards the node *(n +* 1) (i.e., the ground) is:

b.1) —1, for all the conductances which are located on the path between the node *e* and the node *(n +* 1); i.e.,7 = *p(k)* = *p"(e); q* ≥ 0.

b.2) 0, for all the conductances which are not located on the path between the node *e* and the node *(n +* 1); i.e.,7 = *p(k), k* * $p^q(e)$; *q* ≥ 0

where *j, k* = 1, 2, • • • , *n.*

The detailed proof is as follows.

For any node *h* on an *RC* tree the time varying conductance between the nodes *w* and *h* of the adjoint network is denoted by $g_{n,,}(t)$; *w = p(h).* As *t* increases, the voltage at each capacitor in the network *N* approaches the final value M(OO); i.e., ?>„.;,(r) reduces to zero. Therefore, g,,y,(O approaches towards its limit value a„/, i.e.:

$$a_{ul}, = \lim_{t\to\infty} g_{,,}(r) = \frac{\partial i_{wh}(v)}{dv}\Big|_{v=0}. \quad (5.9)$$

Similarly let $C_h(t)$ approach ir,,. Then

$$7T_{,} = \lim_{t\to\infty} C_h(t) = C_{t_i}(u(\infty)). \quad (5.10)$$

Thus for given positive real quantities eO, el, there exists $f_2 \ge 0$ such that:

$$|g_{wh}(t) - \sigma_{wh}| \le e0 \quad \text{and} \quad |C_h(t) - \pi_h| \le el, \quad \text{for } t \ \&\ u.$$

Let C_x denote a diagonal matrix of all the capacitors -*K*,,. Also let G_o denote a conductance matrix of the resistor network whose conductances are given by $a_{u\cdot h}$. Then by choosing an appropriate l_2 we can always choose *tl* and *e3* such that:

$$\|G(t) - G_a\| \le e2 \quad \text{and} \quad \|C(t) - C_T| \le e3. \quad \text{for? } \ge t_2.$$

Equivalent^ | #„,,(T,. - 0) - a„,„ | ≤ ,0, |C,,(T,, - 0) - irj ≤ cl implies

$$\|G(T_{,,} - 6) - GJ\| \le \epsilon 2$$

and

$$\| C(T_{(}. - 0) - cv \| \le «3, \quad \text{forO} \le 0 < (7_{,,} - r_2). \quad (5.11)$$

Now consider a system N_{el}, a linear time invariant /?C tree having constant resistances and capacitances obtained from (5.9) and (5.10) and excited by unit impulse current at the node *e.* The system equations of this limit adjoint *RC* tree (N_{B1}) are

$$C_T\dot{K}(0) + G_aY = 5(6)e,. \quad 0 \le 0 \le (T, - t_2) \quad (5.12)$$

where K(0) is the node voltage vector of $A/_{B1}$ and $\dot{Y}(\theta) = dY(6)/dd.$ Comparing (5.12) with (3.23) and using (5.11). we conclude that there exists a positive real quantity e4 such that:

$$\|X(\theta) - Y(6)\| < e4, \quad 0 \le 0 \le (T_{,,} - i_2).$$

It may be observed that e4 can be made as small as we want by increasing l_2. i.e.. *Y(6)* can be made as much close to *X(d)* as we want.

LetF= -*C^G*,,.

Eventually all the capacitors in N_{fl} and N_{ol} are in the discharging mode as the eigenvalues of both B(0) and *F* arc real distinct, and negative [151, and *Y(d)* approximates *X(d).* Therefore, there exists a 0* > 0 such that:

$$o\text{-}_{Ik}Y_{Ik} \le 0, \quad \text{for}0 \ge 0^*. \quad (5.13)$$

This implies that

$$g_{jk}(T_{,}. - 8)X_{JL} \le 0, \quad \text{for } 0 \ge 0^* \quad (5.14)$$

where

$$X_{ik} = X_{,}(d) \sim X_{(}(0) \quad \text{and} \quad Y_{lk} = Yj(8) - Y_k(d).$$

For any $/_{,,} \le r_{(}$. and $0^* \le (T_{,,} - r_{()})$, let:

$$Q1(\tau_{e} - t_{0}) \doteq \int^{(T_{,} - t_0)} gjdT_r\text{-}e)x_{Jt}de. \quad (5.15)$$

1) For all the conductances which are located on the path between the node *e* and the node *(11+* 1); i.e., *j = p(k) = p"(e); q* ≥ 0, *Ql* (T,, — ?o) represents the charge which has flown through the time varying conductances between the nodes *j* and *k* in the adjoint network N_B towards the ground up to 0 = (T,. — t_0). Thus *Ql* ($T_{(}$, — t_0) will be approaching — 1 by increasing T,..

2) For all the conductances which are not located on the path between the node *e* and the node *(n +* 1): i.e., *j = p(k), k* =£ $p^q(e)$\ *q* ≥ 0, *Q*\(T_C — ?„) represents the remaining charge at 0 = (T_C — t_0) that will flow towards the ground after 0 = (T(, — $/_0$) through the time varying conductance between the nodes *j* and *k* in the adjoint network N_t at 0 = (r(. - r_0). Thus *QKr,, - t0)* can be made infinitesimally small by increasing T_C.

Similarly, let *Q2* be defined as

$$Q2(\tau_e - t_0) \doteq \int_0^{(T_e - t_0)} Q_{ii}X_{jk}\, d\theta. \quad (5.16)$$

From (5.15) and (5.16). for any 0 = 0,:

$$Q\backslash - Ql = \int_0^{\theta_1} [g_{ik}(T_{,}. - 0) - a_{/t}]X_M rf0 + \int_{\theta_1}^{(T_{,} - t_0)} U'MK, - 0) - a_{)k}]X_{jk}dd$$

so

$$|Q1 - Q2| \le \Delta g_1 \int_0^{\theta_1} X_{jk}\, d\theta + \Delta g_2 \int^{(T_{,} - t_0)} X_{ik}\, d\theta$$

where

$$|g_{jk}(T_n - 8) - a_{lk}| \le \Delta g_1, \quad 0 \le 0 \le \theta_1$$

and

$$\left| g_{jk}(T_e - 0) - a_{jk} \right| \leq Ag_2 \leq (a_{\max} - a_m).$$

$$0, \leq 0 \leq T_e - ?_{,,}$$

For a given 6,, Agi can be made infinitesimally small by increasing r,. as can be seen from (5.9). Furthermore, X_{jk} is bounded and infinitesimally small (see (4.7) and [16]) for large *8.* Therefore, $Q_1 - Q_2$ can be made infinitesimally small for a large 0, by increasing T,,. Thus Q_2 approximates Q_1 for large T,,. Since $a_{lk} > 0$:

$$\int_0 X_{jk}\, d\theta \doteq q_a \text{ approaches } \frac{Q_1}{\sigma_{jk}}$$

for large T,.; or:

$$-\int_{\theta^*}^{(T_e - \theta_0)} X_{jk}\, d\theta = \int_0^{\theta^*} X_{jk}\, d\theta - q_o \qquad (5.17)$$

where 0* is as defined in (5.13) and (5.14).

Substituting $\lambda_{jk}^{nr} = [X_{jk}^{nr}(0 - X_{jk}^{nr}(\cdot)]$ for X_{jk} and changing the integrating dummy variable 0 to / = (r(, — *6)* in (5.17), we have

$$-\int_{t_0}^{t^*} \lambda_{jk}^{nr}(t)\, dt = \int_{t^*}^{T_e} \lambda_{jk}^{nr}(t)\, dt - q_o \qquad (5.18)$$

where $I^* = (T_e - 0^*)$. It may be observed that as r_t, increases t^* also increases.

Since $v_{jk}(t)$ reduces to zero as / -» oo and all the /-Kcurves pass through the origin, therefore, we can choose a $I_o \geq 0$ such that $[di_{ik}(v_{jk}(t))/dp_{lk}]$ monotonically reduces to zero for $t > ?_{,,}$ and the following holds:

$$z(\cdot) = \frac{\left[\partial i_{jk}(v_{jk}(t))/\partial p_{jk}\right]}{\left[\partial i_{jk}(v_{jk}(t^*))/\partial p_{jk}\right]} \begin{cases} > 1, & t_0 \leq t \leq t^* \\ < 1, & t^* < t < T_e \end{cases}$$

because

$$\lambda_{jk}^{nr}(t) = \begin{cases} > 0, & t_0 \leq t \leq t^* \\ < 0, & t^* < t < T_e \end{cases}$$

and as T_e increases

$$q_\theta \begin{cases} -1/\sigma_{jk} < 0, & j = p(k) = p^q(e);\ q \geq 0 \\ 0 & j = p(k),\ k \neq p^q(e);\ q \geq 0. \end{cases}$$

Therefore, from (5.18):

$$-\int_{t_0}^{t^*} z(t)\lambda_{jk}^{nr}(t)\, dt > \int_{t^*}^{T_e} z(t)\lambda_{jk}^{nr}(t)\, dt.$$

This leads to

$$\int_{t_0}^{T_e} \left[\partial i_{jk}(v_{jk}(t))/\partial p_{jk}\right]\lambda_{jk}^{nr}(t) < 0.$$

Also:

$$i_{jk}(t) > 0, \qquad \text{for } t > 0$$

and

$$\lambda(t) < 0, \qquad \text{for } 0 < t < t_0.$$

Therefore:

$$\int_0^{T_e} \left[\partial i_{jk}(v_{jk}(t))/\partial p_{jk}\right]\lambda_{jk}^{nr}(t)\, dt < 0.$$

Since $i_{jk}'(T_e) > 0$, it follows from (5.4) that

$$\frac{\partial T_e}{\partial g_{jk}} \leq 0.$$

Remarks

Let the increment in the signal delay at any node be ΔT_e, then from (3.10) we have

$$\Delta T_e = \sum_{k=1}^{n} \frac{\partial}{\partial p_{jk}} \Delta p_{jk} + \frac{\partial}{\partial p_k} \Delta p_k, \qquad j = 1, \cdots (n+1)$$

$$= \sum \frac{\partial T_e}{\partial p_{jk}} \Delta p_{jk} + \sum \frac{\partial T_e}{\partial p_{jk}} \Delta p_{jk} + \sum_{k=1}^{n} \frac{\partial T_e}{\partial p_k} \Delta p_k$$

where $j = p(k)$.

From Corollary 1 and Theorem 2:

a) for small T_e: if $\Delta p_{jk} \geq 0$: $k = p^q(e)$. $\Delta p_{jk} \leq 0$; if $k \neq p^q(e)$, $\Delta p_k \leq 0$ we have $\Delta T_e \leq 0$;

b) for large T_e: if $\Delta p_{jk} \geq 0$; $\Delta p_k \leq 0$ we have $\Delta T_e \leq 0$.

Conditions in (a) suggest that for a given node *e.* the delay time T_e to attain the target voltage would be underestimated if the target voltage is small and i) all the resistors in between the nodes *e* and the source are under estimated, ii) other branch resistors are overestimated, and iii) all capacitors are underestimated. Similarly conditions in (b) suggest that T_e would be underestimated if the target voltage is large and i) all the resistors underestimated and ii) all capacitors are underestimated.

VI. Conclusions

In this paper a new approach is used to study the sensitivity of delay time with respect to a variation in a parameter of resistors/capacitors of a nonlinear monotone *RC* tree. Around $t = 0$, the voltage response rises faster for some nodes and slower for others, if a resistor parameter is varied so as to increase the current through it. This shows that we cannot hope to improve the performance at all nodes at the same time. When the response is close to steady state, the response of all nodes is faster. These results can be used to identify parameters to improve the performance at a critical node of the tree. Also these results can be used to find the bounds on delay time for small and large target voltages at any node on the *RC* tree.

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