

A two-layer model for studying the effect of plasma layer on the delivery of oxygen to tissue using a finite element method

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A mathematical model is described for the transport of oxygen in the systemic capillaries and the surrounding tissue. The model takes into account the molecular diffusion, the convective effect of the blood, the nonlinear effects of oxyhaemoglobin, and the consumption of oxygen in the metabolic process. A two-layer model for the blood consisting of a core of erythrocytes surrounded by a cell-free plasma layer has been considered. A finite element formulation has been given to solve the resulting nonlinear convective-diffusion equations with the physiologically relevant boundary conditions. A fixed point iterative technique is used for the nonlinear terms. It is found that PO_2 (partial pressure of oxygen) in the tissue increases as the core-to-capillary diameter ratio increases. The tissue PO_2 is found to be lower with a heterogeneous model in comparison with a homogeneous model, and thus the analysis shows that the plasma layer obstructs the transport of oxygen from the blood to the tissue. The effect of capillary diameters and core radii on the delivery of oxygen to tissue has also been examined.

Keywords: oxygen transport, mathematical model, finite element method, computer simulation, systemic capillaries, tissue PO_2 , two-phase flow

1. Introduction

When blood flows through a narrow vessel a cell-depleted or cell-free plasma layer has been observed near the wall.¹ There is both theoretical and experimental evidence that the particles near the tube wall tend to migrate toward the axis, the effect being more pronounced in the case of flexible particles.^{2,3} Capillaries where the process of gas exchange takes place are among the smallest vessels (except for collapsed ones) in the circulatory system with diameters ranging from 2.7 to 10 microns.⁴ The diameter of a normal red blood cell (RBC) is approximately 8 microns.⁵ Due to the comparable sizes of the capillaries and red blood cells the latter must travel through capillar-

ies in single file. Further they deform from their biconcave shape to enter the smaller capillaries.^{6,7} Thus blood can no longer be considered as a homogeneous fluid.

A number of mathematical models have been developed to describe the transport of O_2 and other gases in the pulmonary and systemic circulations.⁸⁻¹¹ Reneau, Bruley, and their coworkers¹² extended the classical model of Krogh by including intracapillary diffusion and a nonlinear oxygen dissociation curve. Sharan et al.¹³ have studied O_2 transport in the capillary and tissue in a hyperbaric environment by including the molecular diffusion of O_2 in the capillary and tissue in both radial and axial directions and the saturation of haemoglobin with O_2 . The nonlinear O_2 dissociation curve was linearised to simulate the hyperbaric conditions. Sharan and Selvakumar¹⁴ examined the effects of chemical kinetics on the oxygen delivery to tissue. In all of the foregoing studies the intracapillary description is simplified by assuming that capillary blood is a homogeneous fluid, thereby ignoring the particulate or discrete (two-phase) nature of blood.

The rheological properties of the red cells influence the flow resistance.^{6,15,16} Studies of capillary velocity distribution^{17,18} reveal the heterogeneous effect of the blood, and it can affect the delivery of O_2 to the tissue. The

existence of a cell-free layer near the wall led to the development of a two-phase model for computing the flow in narrow vessels.¹⁹ An axial train model for the blood flow in fine tubes has been developed by Whitmore.⁶ By incorporating the two-phase flow the transport of oxygen in the capillaries has been studied by Kumar.²⁰ However in this study the transport of O_2 in combination with haemoglobin was not considered. Sharan et al.²¹ proposed a two-phase flow model for the oxygenation of blood in the lungs using an axial train blood flow model.

The physiological importance of the resistance of intra-capillary transport has been discussed by Federspiel and Popel.²² They have considered the particulate nature of the blood. In particular, red cells flowing in a single file through capillaries were modelled as evenly spaced haemoglobin containing axisymmetric spheres in a circular tube. They considered the peripheral layer as well as plasma in the gaps between the cells. However, to simplify the analysis, the convection in the plasma was not considered in the transport of oxygen, and thus they did not consider explicitly the hydrodynamic equations to compute flow in the plasma gaps as well as in the peripheral layer. The velocity of red blood cells was specified.

In our earlier studies for the transport of oxygen in the blood flowing through the systemic capillaries the blood has been considered as a homogeneous fluid.^{13,14,23,24} In the present study a more realistic heterogeneous model is developed for the transport of oxygen in the capillary and surrounding tissue by considering an axial train model for the blood, proposed by Whitmore,⁶ consisting of a core of RBC's surrounded by a cell-free plasma layer. Expressions for velocities in the core and the plasma layer from the creeping flow equations are used. Here we neglect the effect of bolus flow developing in the gaps between cells. The nonlinear behavior of oxyhaemoglobin has been incorporated in the model, and a finite element method is used to solve the resulting system of nonlinear partial differential equations. The effect of a cell-free plasma layer on the delivery of oxygen to tissue is then examined.

2. Mathematical formulation

In a capillary the red cells tend to follow each other in a single file with their discoidal surfaces perpendicular to the axis of the capillary. As the flow rate rises, deformation is observed, the cells often becoming thimble shaped with their effective diameter decreasing. Whitmore²⁵ has suggested that a model in this case consists of an axial train of cells moving with their discoidal surfaces normal to the direction of the flow and interspersed with plasma, surrounded by a sleeve of plasma in which shearing takes place. We consider a two-layer model for the blood, consisting of a core of suspended RBC's of uniform haematocrit H_c surrounded by a plasma layer near the wall and a tissue cylinder (Figure 1). When blood flows through the capillary, oxygen diffuses from the core to the tissue through the plasma layer.

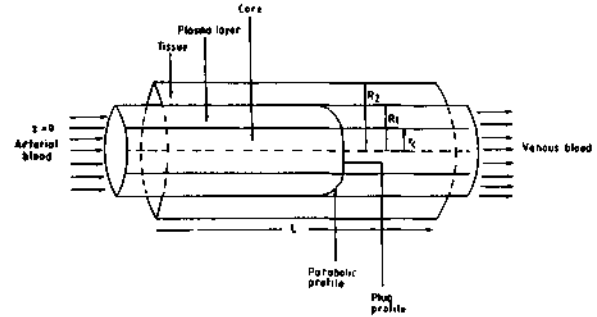


Figure 1. Schematic diagram of the model.

2.1 Governing equation for mass transfer in the capillary

Plasma layer.—Oxygen in the plasma layer is transported by molecular diffusion in both radial and axial directions and by convection. In the steady state a mass balance for O_2 in the plasma layer leads to the following partial differential equation in dimensionless form:

$$\frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial p_1}{\partial r} + \frac{D_{zp}}{D_{rp}} \frac{\partial^2 p_1}{\partial z^2} = P_e w(r) \frac{\partial p_1}{\partial z},$$

$$\gamma < r < 1; \quad 0 \leq z \leq L/R_1 \quad (1)$$

where $P_e = R_1 W_{av} / D_{rp}$ is the Peclet number, p_1 is partial pressure of O_2 in the plasma layer, D_{rp} and D_{zp} are the diffusion coefficients along radial and axial directions in the plasma layer, W_{av} is the average velocity of the blood, R_1 is the radius of the capillary, L is the capillary length, γ is the dimensionless core radius, and w is the velocity in the axial direction. In a dimensionless scheme p_1 is referred to the characteristic partial pressure p_c , r to R_1 , z to L , and w to W_{av} . A symbol with subscript c denotes the corresponding quantity (except for p_2) in the core.

Haemoglobin core.—Oxygen in the core is transported not only through molecular diffusion and convection, as in the plasma layer, but also by convection in combination with haemoglobin. Taking into account these transport mechanisms the steady-state mass balance of O_2 in the core leads to the following dimensionless partial differential equation:

$$\frac{D_{rc}}{D_{rp}} \left(\frac{\partial^2 p_2}{\partial r^2} + \frac{1}{r} \frac{\partial p_2}{\partial r} \right) + \frac{D_{zc}}{D_{rp}} \frac{\partial^2 p_2}{\partial z^2}$$

$$= P_e (1 + \phi(p_2)) w_c \frac{\partial p_2}{\partial z},$$

$$0 < r < \gamma; \quad 0 \leq z \leq L/R_1 \quad (2)$$

where p_2 is the partial pressure of O_2 in the core. To have the same Peclet number in equations (1) and (2) equation (2) has been divided by D_{rp} . The function Φ is

proportional to the slope of the oxygen dissociation curve¹³:

$$\phi(p_2) = \frac{N}{\alpha_c p_c} \frac{\partial \Psi}{\partial p_2} \quad (3)$$

in which N is the O_2 carrying capacity of the blood, α_c is the solubility coefficient of O_2 in the core, and ψ is the fractional saturation of haemoglobin with O_2 . In equation (2) it is assumed that the oxygen and haemoglobin are in chemical equilibrium inside the red blood cells.

Many functional forms can be used to represent ψ in equation (3).²⁶ In the present study we use Hill's equation, and accordingly we have

$$\phi(p_2) = \frac{K_1 p_2^{n-1}}{(1 + K_2 p_2^n)^2} \quad (4)$$

where

$$K_1 = \frac{Nkn p_c^{n-1}}{\alpha_c} \quad (5)$$

and

$$K_2 = k p_c^n$$

in which k and n are Hill's parameters.

2.2. Governing equation in the tissue

The transport of oxygen in the tissue region depends upon molecular diffusion and metabolic consumption of O_2 . In the steady state the material balance of O_2 in the tissue is given by the dimensionless equation⁸:

$$\frac{\partial^2 p'}{\partial r^2} + \frac{1}{r} \frac{\partial p'}{\partial r} + \frac{D'_z}{D'_r} \frac{\partial^2 p'}{\partial z^2} = G, \quad 1 < r < R; \quad 0 \leq z \leq L/R_1 \quad (6)$$

where $G = gR_1^2/(\alpha' D'_r p_c)$ and $R = R_2/R_1$ are dimensionless constants. R_2 is the radius of the tissue cylinder, α' is the solubility of oxygen in the tissue, D'_r and D'_z are the diffusion coefficients of O_2 in the radial and axial directions in the tissue, g is the rate at which oxygen is consumed in the tissue, here taken to be a constant (i.e., zero-order chemical kinetics).

2.3. Boundary and Interface Conditions

Equations (1), (2), and (6) are subject to the following boundary and interface conditions:

(i) Due to symmetry:

$$\text{at } r = 0, \quad \frac{\partial p_2}{\partial r} = 0, \quad 0 \leq z \leq L/R_1 \quad (7a)$$

(ii) The diffusion of oxygen into the tissue is assumed to occur through the capillary-tissue interface only, and there is no transport of oxygen through the outer wall and annular ends of the tissue. Accordingly we have:

$$\text{at } r = R, \quad \frac{\partial p'}{\partial r} = 0; \quad 0 \leq z \leq L/R_1 \quad (7b)$$

$$\text{at } z = 0, \quad \frac{\partial p'}{\partial z} = 0; \quad 1 < r < R \quad (7c)$$

$$\text{at } z = \frac{L}{R_1}, \quad \frac{\partial p'}{\partial z} = 0; \quad 1 < r < R \quad (7d)$$

where R is the dimensionless tissue radius.

(iii) At the entry of the capillary the partial pressure of O_2 is the same as in the arterial blood, i.e.,

$$\text{at } z = 0, \quad \left\{ \begin{array}{l} p_1 = p_{art} \quad \gamma < r < 1 \\ p_2 = p_{art} \quad 0 < r < \gamma \end{array} \right\} \quad (7e)$$

where p_{art} is the dimensionless partial pressure of O_2 in the arterial blood.

(iv) At the exit of the capillary we assume no diffusive flux conditions:

$$\text{at } z = \frac{L}{R_1}, \quad \left\{ \begin{array}{l} \frac{\partial p_1}{\partial z} = 0 \quad \gamma < r < 1 \\ \frac{\partial p_2}{\partial z} = 0 \quad 0 < r < \gamma \end{array} \right\} \quad (7f)$$

(v) The partial pressure of O_2 and its flux across the interface separating the two layers are continuous:

$$\text{at } r = \gamma, \quad \left\{ \begin{array}{l} p_1 = p_2 \\ \frac{\partial p_1}{\partial r} = \frac{\alpha_c D'_{rc}}{\alpha_p D'_{rp}} \frac{\partial p_2}{\partial r} \end{array} \right\}; \quad 0 \leq z \leq L/R_1 \quad (8a)$$

where α_p is the solubility of O_2 in the plasma.

(vi) Since the mass and flux of O_2 across the capillary-tissue interface are continuous we have:

$$\text{at } r = 1, \quad \left\{ \begin{array}{l} p_1 = p' \\ \frac{\partial p_1}{\partial r} = \delta \frac{\partial p'}{\partial r} \end{array} \right\}, \quad 0 \leq z \leq L/R_1 \quad (8b)$$

where

$$\delta = (\alpha' D'_r)/(\alpha_p D'_{rp})$$

2.4. Governing equations for the flow

Due to the small sizes of the capillaries and low flow velocity the Reynolds numbers in the capillary blood flow are very small (10^{-2} to 10^{-3}). The equation of motion for

an incompressible steady flow with a low Reynolds number in the plasma layer reduces to^{3,6}:

$$-\frac{\partial P}{\partial z^*} + \frac{\mu_p}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial W}{\partial r^*} \right) = 0 \quad (9)$$

where W is the velocity in the longitudinal direction, μ_p is the viscosity of the plasma, and P is the hydraulic pressure. r^* and z^* represent the dimensional radial and axial coordinates.

The flow is subject to the following boundary conditions:

(a) No slip condition at the wall of the capillary:

$$\text{at } r^* = R_1, \quad W = 0 \quad (10)$$

(b) Continuity of the velocity at the interface:

$$\text{at } r^* = r_c, \quad W = W_c \quad (11)$$

where W_c is the velocity of the red cell core.

Balancing the shear stress (τ_h) in the core with the hydraulic pressure gradient yields:

$$\tau_h = -\frac{r_c}{2} \frac{\partial P}{\partial z^*} \quad (12)$$

According to Newton's law of viscosity we have:

$$\tau = -\mu_p \frac{\partial W}{\partial r^*} \quad (13)$$

where τ is the shear stress in the plasma layer.

It can be shown that the velocities in the plasma layer and the core satisfying equation (9) and the relations (10)–(13) are given by

$$W = \frac{\Delta P R_1^2}{4L\mu_p} (1 - r^2) \quad (14)$$

$$W_c = \frac{\Delta P R_1^2}{4L\mu_p} (1 - \gamma^2) \quad (15)$$

where ΔP is the difference of the hydraulic pressures between the arterial and venous ends of the capillary. $r = r^*/R_1$ and $\gamma = r_c/R_1$ are the dimensionless radial coordinate and core radius, respectively.

The average velocity (W_{av}) of the blood is obtained by computing the volumetric flow rate by using equations (14) and (15):

$$W_{av} = \frac{1}{2} W_c (1 + \gamma^2) \quad (16)$$

From equations (14)–(16), we have

$$W_c = \frac{2W_{av}}{1 + \gamma^2} \quad (17)$$

$$W(r) = \frac{2W_{av}}{1 - \gamma^4} (1 - r^2)$$

The relationship between the core haematocrit (H_c) and the haematocrit of the blood (H_b) is obtained by the procedure outlined in Lih²⁷ (page 403) and is given by:

$$\frac{H_c}{H_b} = \frac{1 + \gamma^2}{2\gamma^2} \quad (18)$$

We wish to point out that the velocities in the core and plasma layer given in equation (17) are to be nondimensionalized before using in the mass-transfer equations (1) and (2).

To find an analytical solution in the tissue region we have neglected the axial diffusion. The role of axial diffusion in the capillary is negligibly small, and it can also be ignored. By neglecting the axial diffusion in the capillary region the elliptic partial differential equation (PDE) in equation (1) becomes a parabolic PDE. However it is relatively easier to deal with the elliptic PDE than the parabolic PDE using the finite element method. Thus we have retained the axial diffusion in the capillary region and neglected it in the tissue region.

3. Analytical solution in the tissue region

By neglecting the axial diffusion in the tissue equation (6) becomes

$$\frac{\partial^2 p'}{\partial r^2} + \frac{1}{r} \frac{\partial p'}{\partial r} = G \quad (19)$$

Integrating the equation (19) and applying the boundary condition (7b), the O_2 tension gradient is given by

$$\frac{\partial p'}{\partial r} = \frac{G}{2} \left(r - \frac{R^2}{r} \right) \quad (20)$$

Integrating equation (20) and using the boundary condition (8b) we obtain:

$$p'(r, z) = p_1(1, z) + \frac{G}{2} \left(\frac{r^2 - 1}{2} - R^2 \ln(r) \right) \quad (21)$$

Notice that the interface condition (8b) on the capillary side is known explicitly through equation (20) and thus equations (1) and (2) in the plasma and core regions can be solved independently of the tissue region. Since equations (1) and (2) are nonlinear and are not amenable to analytical treatment we solve them by the method of finite element.

4. Finite element formulation

Equations (1) and (2) can be put in the form:

$$A \left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} \right) + B \frac{\partial^2 p}{\partial z^2} = P_e V(p) \frac{\partial p}{\partial z} \quad (22)$$

where

$$A = 1, \quad B = D_{zp}/D_{rp}, \quad \text{and} \quad V = w(r) \quad (23)$$

for equation (1) in the plasma layer and

$$A = D_{rc}/D_{rp}, \quad B = D_{zc}/D_{rp}, \quad \text{and} \\ V = (1 + \Phi(p))W_c \quad (24)$$

for equation (2) in the core.

We divide the capillary region enclosed by a surface S into a finite number of elements. We have taken M_1 elements in the core region and M_2 elements in the plasma layer (Figure 2). For the finite element formulation of equation (22) we use Galerkin's method, which is a particular form of method of weighted residuals.

For using Galerkin's method we assume that the unknown field variable (i.e., the partial pressure of O_2) within a typical element can be approximated by²⁸:

$$p^e(r, z) = \sum_{i=1}^m N_i(r, z) p_i = [N]\{p\} \quad (25)$$

where N_i are interpolation or shape functions defined over an individual element, m is the number of nodes per element, and p_i are the unknown nodal values of the field variable.

Using Galerkin's procedure to equation (22) at node i of an element²³ we get at node i :

$$\int_{V^e} \left(A \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial r} P_j + B \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} P_j \right) dV^e \\ + \int_{V^e} P_e N_i V \frac{\partial N_j}{\partial z} P_j dV^e \\ = \int_{S^e} N_i \left(A \frac{\partial N_j}{\partial r} P_j n_r + B \frac{\partial N_j}{\partial z} P_j n_z \right) dS^e, \\ i = 1, 2, \dots, m \quad (26)$$

where V^e and S^e are the volume and surface of the element e under consideration, and n_r and n_z represent

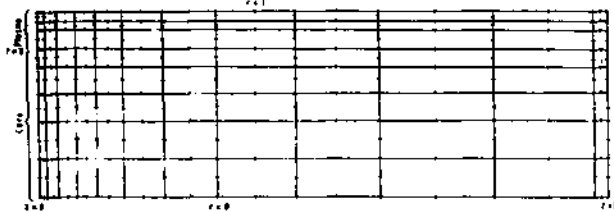


Figure 2. Discretization of the capillary.

the radial and axial components of the outward unit normal to the surface segment dS^e .

The relation (26) will give rise to m equations. This set of equations is the characteristic set that represents the properties of the individual element and can be written in the standard matrix form as:

$$[K]^e \{p\}^e = \{R\}^e \quad (27)$$

with

$$[K]^e = [K_1]^e + [K_2]^e \quad (28)$$

where

$$K_{1ij} = \int_{V^e} \left(A \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial r} + B \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) dV^e \quad (29)$$

$$K_{2ij} = \int_{V^e} P_e N_i V \frac{\partial N_j}{\partial z} dV^e \quad (30)$$

$$R_i = \int_{S^e} N_i \left(A \frac{\partial N_j}{\partial r} n_r + B \frac{\partial N_j}{\partial z} n_z \right) P_j dS^e \quad (31)$$

Here we are using isoparametric curvilinear quadratic elements for dividing the domain into a finite number of elements of the concentration field. The evaluation of the element equations (27) involves the evaluation of the integrals, on the right-hand side of equations (29)–(31), over an element. The integrals are evaluated numerically using a 3×3 Gaussian quadrature formula²⁸ for curvilinear quadrilateral elements.

Since the characteristic matrix $[K]^e$ (equation [28]) is not symmetric the element matrices are to be assembled in full to get the complete set of equations. This requires a very large computer storage. Here we use a frontal technique^{20,23} to assemble the element equations. The surface integral terms in equation (31) will have nonzero values only for elements whose boundaries are part of the domain boundary S and thus offer a convenient way to incorporate the boundary conditions. After substituting the boundary condition the complete system of algebraic equations is solved for the nodal values p_i . The element equations (27) are nonlinear because V is a nonlinear function of nodal values (also, see equations [22]–[24]). Thus on assembling element matrices we get a nonhomogeneous system of nonlinear algebraic equations.

This needs an iterative technique for the solution. We use a fixed point iterative technique²³ in which the nonlinear terms are evaluated at the inlet partial pressure. With this approach the system is linearised and is solved for p . The computed value of p is used to reevaluate the nonlinear terms, and the system is again solved for p . We continue this process until

$$\sum_{i=1}^{m_i} \|p_i^{r+1} - p_i^r\| < \epsilon$$

where r represents the r th iteration, m_i is the total number of nodes in the solution domain and ϵ is the error of tolerance.

5. Parameters of the model

The values of the parameters taken for the computation are given in Table 1 and are taken from Reneau et al.²⁹ and Sharan et al.²³ We have taken the value of core radius (γ) from the in vitro studies of Seshadri et al.³⁰

The diffusion coefficient of O₂ in the plasma (D_p) is computed from the corresponding diffusion coefficient in the blood (D_b) of haematocrit H_b by using the formula³¹:

$$\frac{D_p}{D_b} = \frac{1.58 + 0.64H_b}{1.58 - 0.78H_b} \quad (34)$$

Once the diffusion coefficient of O₂ in the plasma layer is computed the corresponding coefficient in the core will be obtained by replacing D_b and H_b with D_c and H_c in relation (32) while using equation (18) to compute H_c .

The haematocrit of the blood (H_b) is taken to be 40%. The characteristic partial pressure p_c is taken 100 mmHg. PO₂ at the entry is chosen to be 95 mmHg, corresponding to the value in the arterial blood. The length of the capillary is assumed to be 60 times its radius. The value of the Hill parameters k and n are taken from Sharan and Singh.³² The calculated value of the Peclet number P_e is 0.87.

Here we have taken 96 elements (i.e., 60 elements in the core region and 36 in plasma region) with 329 nodes. However the test runs were made for a smaller and larger number of elements in the domain. We have observed that 96 elements are the optimum. Also we have analyzed the computational results on the basis of physiological information available in the literature, and the results are found to be physiologically consistent.

Since the isoparametric curvilinear quadratic elements are adopted we have taken the eight nodes per element.²⁸

6. Results and discussion

We have compared the results obtained from the finite element method with those from the finite difference method based on a line iterative approach.¹⁴ Figure 3

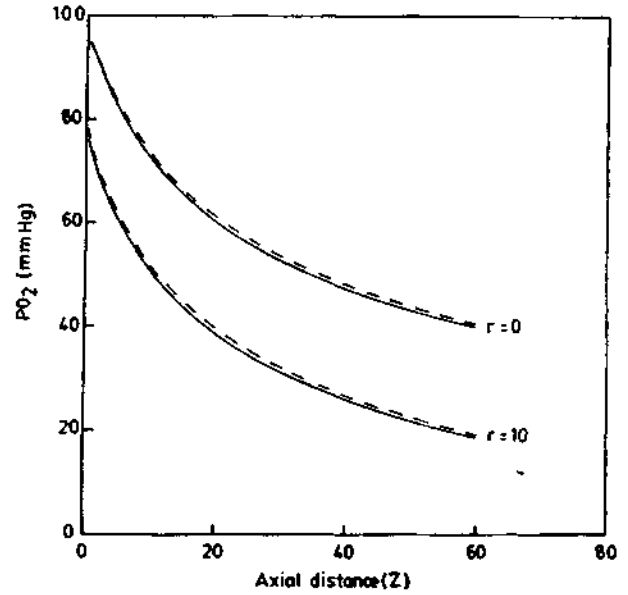


Figure 3. Comparison of the computed PO₂ obtained from the finite element method (—) with that from finite difference method (---) based on a line iterative approach for $\gamma = 1$. Axial distance z is in microns.

reveals that there is a close agreement between the computational results for $\gamma = 1$ obtained from the finite element and finite difference techniques.

Figure 4(a) represents PO₂ in the capillary and tissue for different dimensionless core radii γ . The graph shows that PO₂ in the capillary and tissue decreases as z increases for a given γ . As γ increases, PO₂ in the core decreases, whereas it increases in the plasma layer and in the tissue. This is because of the fact that as γ increases the speed of the core decreases, whereas it increases in the plasma layer (see equations [17]). Thus O₂ in the core decreases because of reduced convection, and it increases in the plasma layer due to enhanced convection. Further the amount of O₂ diffused from the core to the tissue increases with a decrease in the thickness of the cell-depleted (plasma) layer due to the smaller distance covered by O₂ through plasma in the diffusion mechanism. Since PO₂ at the plasma-tissue interface increases as the plasma layer becomes thinner (as γ increases) it is obvious that tissue PO₂ will increase. $\gamma \rightarrow 1$ implies the homogeneous blood filling the entire capillary, and we recover the homogeneous model concentration profiles obtained in an earlier study.²³ Also it is seen from Figure 4(a) that tissue PO₂ in a homogeneous model is higher in comparison to the two-phase model. Thus the analysis using the two-layer model shows that the plasma layer obstructs the delivery of O₂ from the blood to the tissue.

Figure 4(b) represents the axial variation of PO₂ in the capillary and tissue for the capillaries of different diameters and core radii. The data for the capillary diameter and the corresponding core radius have been taken from the in vitro studies of Seshadri et al.³⁰ Figure 4(b) shows that PO₂ in the capillary blood decreases as the radius of the capillary decreases, whereas tissue PO₂ increases. This is because of the fact that a smaller distance is

Table 1. Values of the parameters used in the computation

Parameter	Units	Value
D_1	cm ² /sec	1.12×10^{-5}
D_2	cm ² /sec	1.12×10^{-5}
D'_1	cm ² /sec	1.7×10^{-5}
R_1	cm	3.25×10^{-4}
R'_2	cm	3.25×10^{-3}
W_{av}	cm/sec	0.03
g	mole/cm ³ ·sec	3.72×10^{-8}
α	mole/cm ³ ·mmHg	1.527×10^{-9}
α'	mole/cm ³ ·mmHg	1.295×10^{-9}
n		2.6472
N	mole/cm ³	9.1×10^{-6}
k	(mmHg) ⁻ⁿ	1.73×10^{-4}
γ		0.8

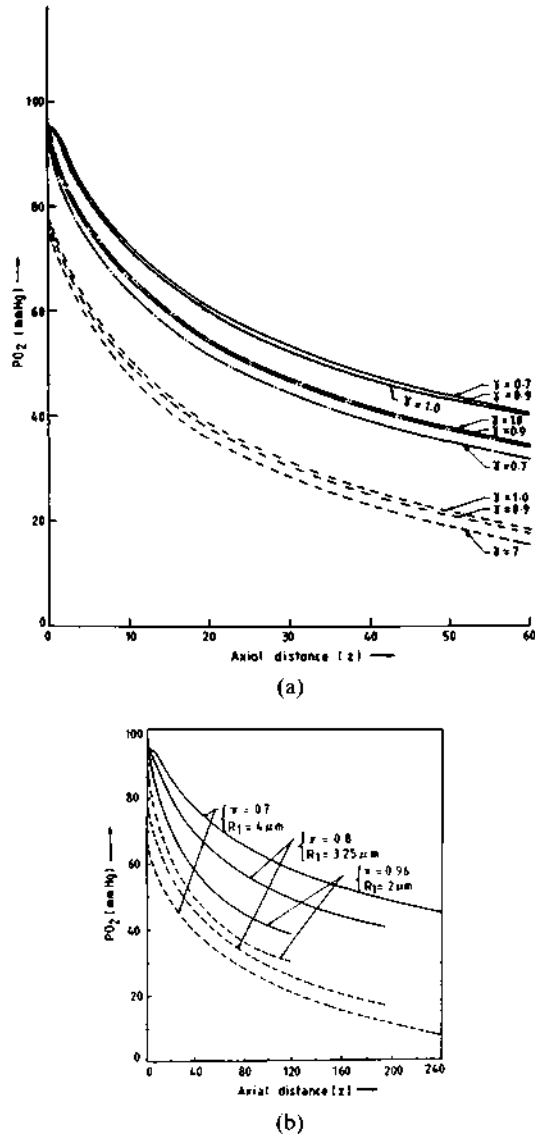


Figure 4. (a) The axial variation of oxygen partial pressure in the capillary at $r=0$ (—), at the interface (---), and in the tissue at $r=10$ (-·-) with core radius (γ). Radial distance r and axial distance z are referred to the capillary radius ($R_1 = 3.25 \times 10^{-4}$ cm). (b) The axial variation of oxygen partial pressure in the capillary (—) and in the tissue (-·-) for different capillary radius (γ). Axial distance z is in microns.

traversed by oxygen in the diffusion mechanism. Hence the tissue PO_2 is lower in the capillary of a larger cross-section.

The model depends on a number of parameters such as PO_2 in the arterial blood, diffusion coefficients, Peclet number, and the diffusive flux of O_2 (i.e., $F = \partial p_1 / \partial r = \delta(\partial p' / \partial r)$ in equation [8b]) from the capillary. The diffusive flux F essentially depends on the metabolic rate.²³ The effect of the individual parameters on oxygen delivery to tissue has been studied in our earlier work.²³ We have examined it in a similar way. The qualitative nature of the results remains the same.

7. Conclusions

We have developed a mathematical model for the transport of oxygen in the capillary and tissue. A nonlinear function for the saturation of haemoglobin with oxygen has been used to represent the experimentally observed sigmoidal oxygen dissociation curve. A finite element formulation has been given to solve the resulting nonlinear convective diffusion equation. A fixed point iterative technique is used for nonlinear terms.

In the present study we have used an axial train model for the blood, proposed by Whitmore,⁶ consisting of a core of RBC's surrounded by a cell-free plasma layer near the wall. The analysis shows that the plasma layer obstructs the transport of O_2 from the blood to the tissue.

When $\gamma \rightarrow 1$ the two-layer model reduces to the homogeneous model. The model is based on a number of simplifying assumptions. It makes no allowance for the bolus flow developing in the gaps between the cells.³ The bolus flow could be as important as the peripheral layer for improving the delivery of O_2 to the tissue. We have assumed that the chemical reactions of haemoglobin with oxygen in the blood are instantaneous. However the nonequilibrium kinetics of O_2 with haemoglobin in the blood plays an important role in the delivery of oxygen to tissue.¹⁴ Sharan et al.³³ have shown theoretically that a considerable amount of O_2 diffuses through the walls of precapillary vessels. The interaction of O_2 and CO_2 in the blood is also important because the concentration and carriage of one gas affects the transport of the other. Thus the present model can be improved by considering:

- (i) the bolus flow;
- (ii) the interaction of O_2 and CO_2 in the blood;
- (iii) the chemical reactions of the gases in the blood¹⁴;
- (iv) the nonlinear metabolism in the tissue;
- (v) the heterogeneity of the capillary network³⁴; and
- (vi) the transport through pre- and postcapillary vessels.^{33,35,36}

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