



DYNAMIC INTERACTION BETWEEN BOUNDARY LAYER FLOW AND GROUNDWATER FLOW SEPARATED BY A POROUS BED

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Abstract—This paper examines the problem of dynamic interaction between a steady-state axisymmetric boundary layer flow over a fluid-saturated porous bed and a uniform Darcy flow occurring in the bed. The analysis is based on the selection of an appropriate boundary condition to account for the existence of tangential slip at the permeable interface separating the two flows. Numerical solutions are presented for different values of permeability parameter K and the characteristic Reynold number Re of the potential flow impinging on the porous bed. It is shown that when $K^2 Re < 1$, there exists a thin sublayer attached to the bed in which the normal velocity of the fluid is directed upwards. Response of pressure distribution in the boundary layer and shear stress on the bed to variations in K and Re has also been examined.

INTRODUCTION

There are numerous investigations [1-12] dealing with the study of flows through and past porous media on account of its many useful applications in several practical fields such as groundwater hydrology, soil mechanics, biophysical flows, petroleum extraction and irrigation engineering. The analysis of such flows is based on the use of an appropriate slip boundary condition proposed by Beavers and Joseph [3], to give a realistic description of the tangential velocity component at the permeable surface. Subsequent experiments [4-6] dealing with flows in channels with porous walls gave results which agreed well with the theoretical predictions obtained by employing this slip boundary condition. Later Saffman [8] provided a sound theoretical justification for the validity of this boundary condition by an analytical derivation based on the idea of a limiting case of a step function distribution of permeability and porosity of the porous matrix. In recent papers, Mehta [9-11] and Mehta and Narasimha Rao [12] have studied the effect of slip at permeable boundaries on steady and transient flows in cavities of circular and rectangular cross section embedded in a non-erodible porous medium.

The aim of this paper is to study dynamic interaction between an axisymmetric boundary layer flow and a uniform Darcy flow separated by a permeable interface. The analysis is based on the boundary condition proposed by Beavers and Joseph [3].

FORMULATION AND BASIC EQUATIONS

Let an axisymmetric column of frictionless potential flow arriving from the z -axis impinge on a flat fluid-saturated porous bed of uniform permeability k occupying the half space $z \leq 0$. After it comes in contact with the plane $z = 0$, the viscosity of the fluid becomes operative. The fluid in the viscous boundary layer flows radially outwards in all directions while the potential flow slides over it. Unlike in the case of an impermeable bed [13] where the fluid will adhere to the bed, a suitable boundary condition for the tangential (radial) component of velocity has to be selected to account for the tangential slip at permeable surface. On account of seepage and transfer of momentum from the boundary layer, a Darcy flow described by

$$u_d = -\frac{k}{\mu} \frac{\partial p}{\partial r}, \quad w_d = -\frac{k}{\mu} \frac{\partial p}{\partial z} \quad (1)$$

is generated just below the interface $z = 0$ (see Fig. 1).

Here u_d, w_d denote the radial and axial velocity components of Darcy flow at the point (r, z) , p , the fluid pressure and μ , the coefficient of dynamic viscosity of the fluid.

The potential flow far away from the bed is chosen in the form

$$u_\infty = \frac{r}{a} u_c, \quad w_\infty = -2 \frac{z}{a} u_c \tag{2}$$

where u_c denotes characteristic velocity.

Note that $u_\infty, w_\infty = u_c, -2u_c$ at the point (a, a) . We now introduce

$$Re = \frac{a u_c}{\nu}$$

as the characteristic Reynold number of the potential flow impinging on the bed; here ν denotes the coefficient of kinematic viscosity of the fluid.

In the potential flow, the pressure p_∞ at an arbitrary point (r, z) is given by the Bernoulli equation

$$p_0 - p_\infty = \frac{1}{2} \rho (u_\infty^2 + w_\infty^2) = p_c (R^2 + 4Z^2) \tag{3}$$

where $p_c = (\rho u_c^2)/2$ is the characteristic pressure, p_0 is the pressure at the point $(0, 0)$, ρ , is the fluid density and $R = r/a, Z = z/a$ are dimensionless radial and axial distance coordinates, respectively.

In the boundary layer region, the flow and pressure fields are assumed to be given by

$$u = Rf'(Z)u_c, \quad w = -2f(Z)u_c \tag{4}$$

$$p_0 - p = p_c [R^2 + 4F(Z)] \tag{5}$$

where f and F are yet unknown dimensionless velocity and pressure functions and the prime denotes differentiation with respect to Z .

Introducing dimensionless velocity components and pressure field defined by

$$U = \frac{u}{u_c}, \quad W = \frac{w}{u_c}, \quad P = \frac{p}{p_c}$$

the potential flow and the boundary layer flow can then be described in dimensionless form by

$$U_\infty = R, \quad W_\infty = -2Z, \quad P_0 - P_\infty = R^2 + 4Z^2 \tag{6}$$

and

$$U = Rf'(z), \quad W = -2f(Z), \quad P_0 - P = R^2 + 4F(Z) \tag{7}$$

It can be easily verified that the equation of continuity

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

is identically satisfied by the flow field (4) or (7).

The two momentum equations for axisymmetric flow in the radial and axial directions,

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right] \tag{8}$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] \tag{9}$$

together with suitable boundary conditions are then sufficient to determine the two functions f and F .

Equations (8) and (9), on using equation (7), yield the following system of equations for f

and F ,

$$\frac{1}{\text{Re}} f''' + 2ff'' - f'^2 + 1 = 0 \quad (10)$$

$$F' = \frac{1}{\text{Re}} f'' + 2ff' \quad (11)$$

Since the Darcy flow occurring just below the plane $Z = 0$ is determined solely from pressure gradient, its dynamic interaction with the boundary layer flow on the bed is studied by invoking the continuity of the radial and axial pressure gradients at every point on the interface $Z = 0$, that is,

$$\left. \begin{aligned} \frac{\partial P}{\partial R}(R, 0-) &= \frac{\partial P}{\partial R}(R, 0+), & R \geq 0 \\ \frac{\partial P}{\partial Z}(R, 0-) &= \frac{\partial P}{\partial Z}(R, 0+), & R \geq 0 \end{aligned} \right] \quad (12)$$

These conditions imply the continuity of the pressure at $Z = 0$, that is,

$$P(R, 0-) = P(R, 0+) \quad (13)$$

On using equations (1), (5) and (12) the dimensionless velocity components of Darcy flow just below the plane $Z = 0$ can be written as

$$U_d = K^2 \text{Re } R, \quad W_d = 2K^2 \text{Re } F'(0) \quad (14)$$

where $K = \sqrt{k}/a$ has been introduced as the dimensionless permeability parameter.

Following Beavers and Joseph [3], the tangential slip at the permeable boundary $Z = 0$ is accounted for by using the following boundary condition satisfied by the radial (tangential) component of velocity,

$$\frac{\partial u}{\partial z} = \frac{\alpha}{\sqrt{k}} (u - u_d) \quad \text{at } z = 0 \quad (15)$$

which, in non-dimensional form, can be written as

$$f'(0) = K^2 \text{Re} + \frac{K}{\alpha} f''(0) \quad (16)$$

Here α is a dimensionless constant depending on the structure of the porous medium.

The continuity of the axial (normal) component of velocity at $Z = 0$

$$W(R, 0-) = W(R, 0+) \quad (17)$$

yields

$$f(0) = -K^2 \text{Re } F'(0) \quad (18)$$

On using equations (11) and (16), equation (18), leads to the boundary condition

$$f(0) = \frac{-K^2 f''(0)}{1 + 2K^4 \text{Re}^2 + 2 \frac{K^3}{\alpha} \text{Re } f''(0)} \quad (19)$$

Since $U \rightarrow U_\infty$ as $Z \rightarrow \infty$, f must also satisfy the asymptotic condition

$$f' \rightarrow 1 \quad \text{as } Z \rightarrow \infty \quad (20)$$

From equation (5), we observe that F satisfies the condition

$$F(0) = 0 \quad (21)$$

Also, since $P \rightarrow P_\infty$ as $Z \rightarrow \infty$, the pressure function $F(Z)$ should behave like Z^2 for large Z .

This asymptotic behaviour of F observed during the computations was used to check the accuracy of numerical results presented in this paper.

The flow field in the boundary layer is completely known by numerically solving equation (10) together with boundary conditions (16), (19) and (20). Once f is known, the pressure function F is found by numerically solving equation (11) with condition (21).

For large values of Re , the problem described by equations (10) and (11) is one of singular perturbation. In this paper, results are presented only for moderate values of Re . In the particular case $K=0, Re=1$ of this study, the results are found to be in complete agreement with those for the problem of axisymmetric stagnation point flow over an impermeable horizontal plane [13].

NUMERICAL RESULTS

The constant α appearing in the boundary conditions (16) and (19) was assigned the value unity during the computations. The main results of this study are summarized in Figs 2-7.

It was found during the computations that the value of $f''(0)$ came out to be zero for every pair of values of K and Re satisfying the relation $K^2 Re = 1$. Furthermore, it can be readily verified that the equations for f and F subject to their respective boundary conditions, on setting $K^2 Re = 1$ and $f''(0) = 0$, admit the solutions

$$f(Z) = Z, \quad F(Z) = Z^2 \tag{22}$$

thereby, giving the potential flow field

$$U = R, \quad W = -2Z, \quad P_0 - P = R^2 + 4Z^2$$

as described by equation (6)

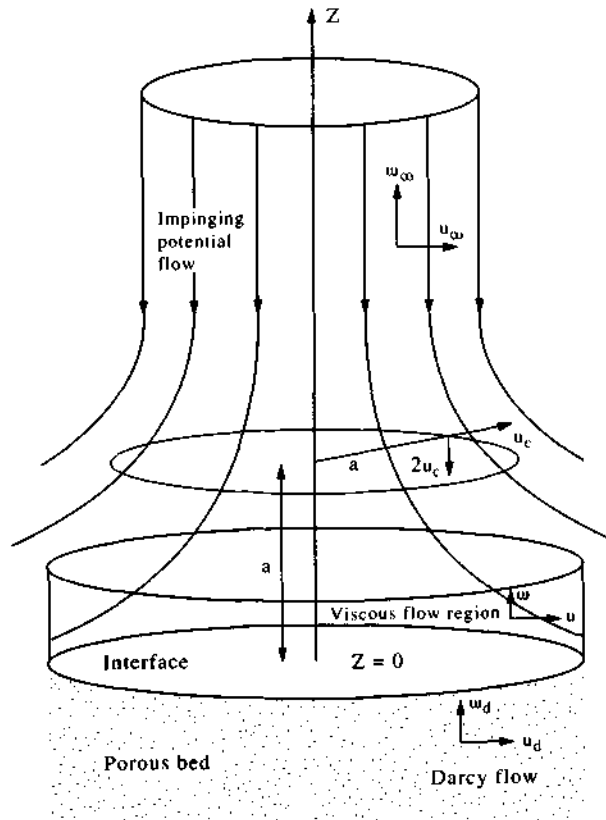


Fig. 1. Physical model.

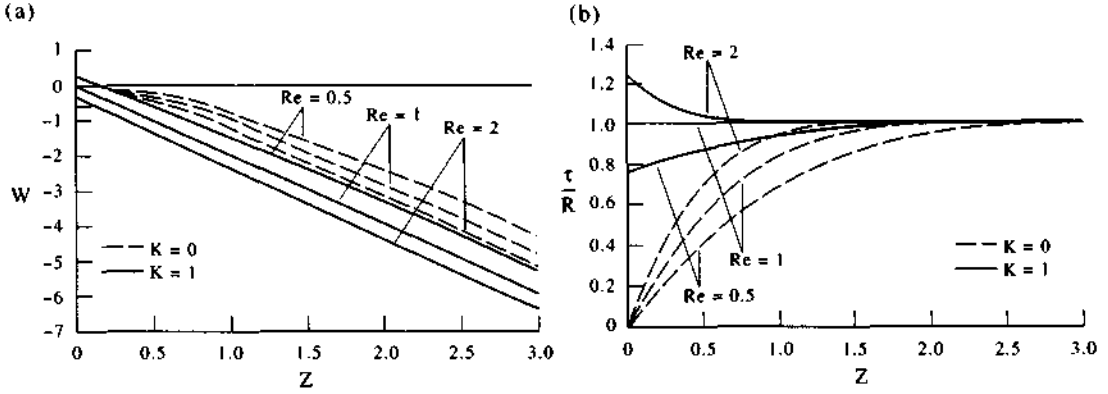


Fig. 2. (a) Axial velocity profiles for different values of permeability parameter K and Reynold number Re . (b) Radial velocity profiles for different values of K and Re .

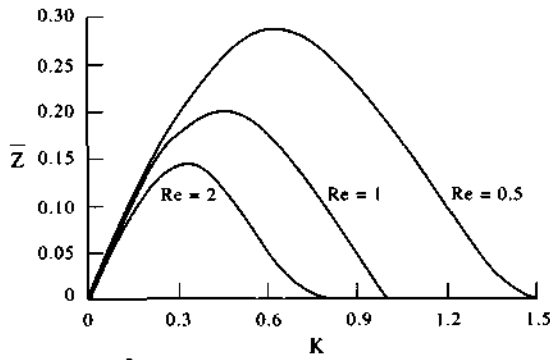


Fig. 3. Variation of the location \bar{Z} of the point of flow reversal in normal velocity with permeability parameter K for $Re = 0.5, 1, 2$.

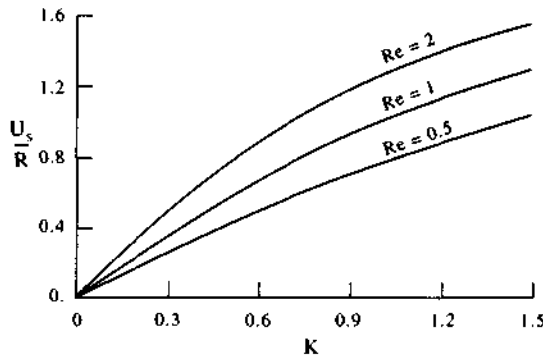


Fig. 4. Variation of the tangential (radial) slip velocity at the bed with K for $Re = 0.5, 1, 2$.

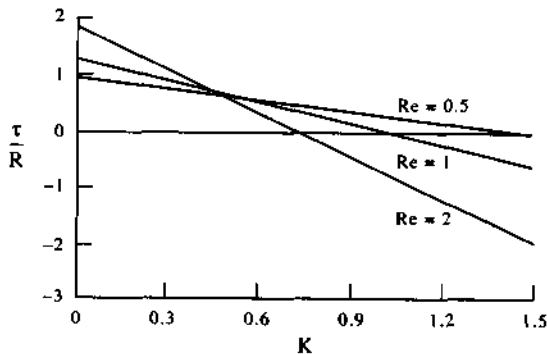


Fig. 5. Response of shear stress on the bed to variation in K for $Re = 0.5, 1, 2$.

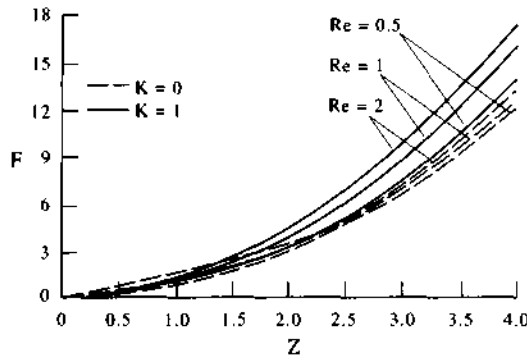


Fig. 6. Pressure distribution for $K = 0, 1$ and $Re = 0.5, 1, 2$.

This implies that the potential flow will extend right up to the interface $Z=0$ under experimental conditions when the permeability parameter K and characteristic Reynolds number Re satisfy the relation $K^2 Re = 1$.

Axial velocity profiles for different values of K and Re are exhibited in Fig. 2(a). For $K^2 Re > 1$, the normal velocity is directed downwards right up to $Z = 0$ but for $K^2 Re < 1$, there is a thin sublayer attached to the bed in which the normal velocity is directed upwards. Also W vanishes at $Z = 0$ when $K^2 Re = 1$. An increase in the values of the parameters K and Re causes an increase in normal velocity.

Figure 2(b) shows that at a given radial location R , the radial velocity $U \propto R$ depending on whether $K^2 Re \cong 1$.

Figure 3 brings out the effect of permeability parameter K on the location \bar{Z} of the point of reversal in the direction of normal velocity. Computations showed that $W = -2 f(Z) \cong 0$ according as $Z \cong \bar{Z}$. The point of flow reversal in the normal velocity exists only when $K^2 Re < 1$.

Figure 4 gives the plot of tangential (radial) slip velocity $U_s/R = f'(0)$ with K for different values of Re . It is seen that at any radial location R , U_s increases with an increase in K and Re .

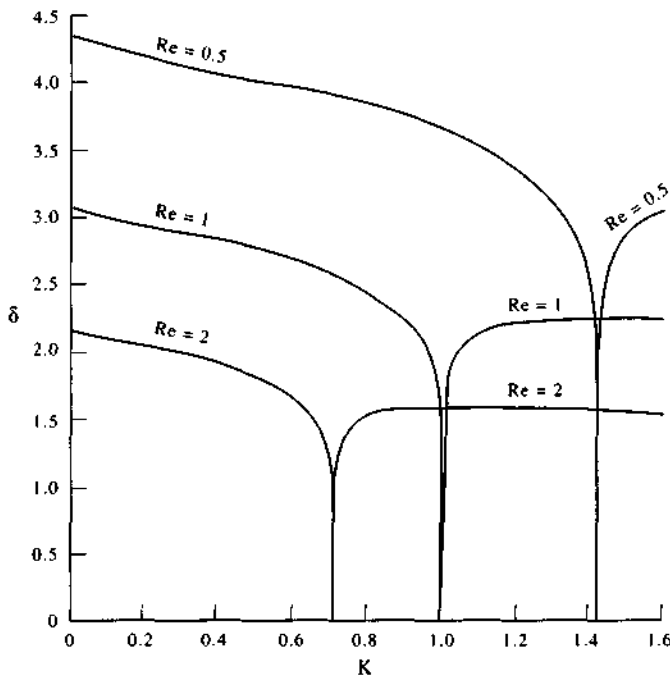


Fig. 7. Variation of boundary layer thickness δ with K for $Re = 0.5, 1, 2$.

The shear stress $\bar{\tau}$ at location R on the bed is given by

$$\bar{\tau} = \mu \left. \frac{\partial u}{\partial z} \right|_{z=0}$$

The non-dimensional shear stress τ , defined by $\tau = \bar{\tau}/(\mu u_c/a)$ can be expressed as

$$\tau = \left. \frac{\partial U}{\partial Z} \right|_{z=0} = Rf''(0) \quad (23)$$

It is observed from Fig. 5 that $\tau \cong 0$ depending on whether $K^2 \text{Re} \cong 1$. It is interesting to note that corresponding to a bed of prescribed permeability K , there exists a unique potential flow with $\text{Re} = 1/K^2$ for which the shear stress will vanish at all points on the bed.

The mass of fluid flowing radially outwards per unit time per unit area of the curved surface of a cylinder of radius r and height z is given by

$$m = \frac{1}{2\pi rz} \int_0^z 2\pi r \rho u(r, z) dz$$

Dimensionless radial mass flux M is then given by

$$M = \frac{m}{\rho u_c} = \frac{R}{Z} [f(Z) - f(0)] \quad (24)$$

Note that

$$\lim_{Z \rightarrow 0} M = Rf'(0)$$

The mass of fluid flowing across per unit area of the bed in unit time is given by

$$n = \frac{1}{\pi r^2} \int_0^r 2\pi r \rho w(r, z) dr$$

Dimensionless normal flux N can then be written as

$$N = \frac{n}{\rho u_c} = -2f(0) \quad (25)$$

From equations (24) and (25), it is seen that M , N satisfy the relation

$$2MZ - NR = 2Rf(Z) \quad (26)$$

Equation (7) together with the result displayed in Fig. 6 implies that the pressure P in the boundary layer decreases with increasing permeability (K) and increasing Reynold number.

It is observed from Fig. 7 that for a given value of Re , as K increases from the value zero, the boundary layer thickness δ first decreases until it vanishes for $K = 1/\sqrt{\text{Re}}$ and then with further increase in K , it increases until it asymptotically attains a constant value. As expected from physical considerations, δ is found to decrease with an increase in Re .

CONCLUSIONS

The slip boundary condition proposed by Beavers and Joseph [3] has been employed to examine the dynamic interaction between a boundary layer flow and ground water flow separated from each other by a permeable bed. The permeability (K) of the porous bed and the characteristic Reynold number Re of the impinging potential flow are identified as the two physical parameters for complete description of the flow studied in this paper. It is found that: (i) the shear stress at every point of the bed vanishes when $K^2 \text{Re} = 1$; and (ii) under experimental conditions corresponding to $K^2 \text{Re} < 1$, there exists a thin sublayer attached to

the bed in which the normal component of fluid velocity is directed upwards, that is in the direction opposite to that of the impinging potential flow.

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NOMENCLATURE

a = Characteristic length
 f, F = Dimensionless velocity and pressure function
 k = Permeability
 K = Dimensionless permeability parameter
 m, n = Radial and axial mass flux
 M, N = Dimensionless radial and axial mass flux
 p = Fluid pressure
 p_c = Characteristic pressure
 \bar{P} = Dimensionless pressure
 r, z = Radial and axial distance coordinates
 R, Z = Dimensionless radial and axial coordinates
 Re = Characteristic Reynold number
 u, w = Radial and axial (normal) velocity components
 u_c = Characteristic velocity
 U, W = Dimensionless radial and axial velocity components
 \bar{Z} = Location of point of flow reversal

Greek symbols

α = Dimensionless constant introduced in equation (15)
 δ = Dimensionless boundary layer thickness
 μ, ν = Coefficients of dynamic and kinematic viscosity
 ρ = Fluid density
 $\bar{\tau}$ = Shear stress on the bed
 τ = Dimensionless shear stress

Subscripts

∞, d = Refer to quantities in potential flow and Darcy flow
 s = Refers to slip at the porous bed