SOME EXISTENCE AND REGULARITY RESULTS FOR NONLOCAL ELLIPTIC EQUATIONS

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by

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Dedicated To My Family

Certificate

This is to certify that the thesis entitled **"Some Existence and Regularity Results for Nonlocal Elliptic Equations"** submitted by **Divya Goel** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

The main topic of the thesis is the study of nonlocal elliptic partial differential equations. In recent decades, the interest in nonlocal PDEs got a huge growth because of its ample amount of applications in many branches of engineering, biology, Physics, and so forth. Nonlocal operators have the characteristic of capturing the long range dynamics. The thesis is divided into five chapters.

In Chapter 1, we give a brief survey, preliminaries used in subsequent chapters and organization of thesis which gives a summary to the contribution of our work.

In Chapter 2, we investigate the spectral properties of the nonlocal operators. We established the existence of second eigenvalue and first nontrivial curve in the Fučik spectrum of different types of nonlocal operators by means of variational methods. Next, we develop the sharp lower bounds for the first and second eigenvalue like "Faber-Krahn inequality" and "Nonlocal Hong-Krahn-Szego" inequality."

In Chapter 3, we present the effect of the geometry of the domain on the existence and multiplicity of solutions to Choquard equations involving critical nonlinearity. We have established the global compactness result, which is of independent interest. We first prove the Coron type result for the critical Choquard problem. Next, We adapt the tools of critical point theory, degree theory, and variational methods to prove the existence of four positive solutions to a nonhomogenous problem. Further, we have also discussed the convex problem and prove that the peculiarity of the topology of domain yields a lower bound on the number of positive solutions.

In Chapter 4, we demonstrate the regularity of solutions for a class of doubly nonlocal problems. Precisely, we have studied a problem involving fractional Laplacian with a general critical Choquard non linearity. We develop a nonlocal Brezis-Kato techniques to obtain the uniform bound. We concluded the uniform a-priori bound of the weak solutions by developing the Brezis-Kato technique in the doubly nonlocal framework even in case of singular nonlinearity. We also have Sobolev versus Hölder weighted minimizers result for the problem.

In Chapter 5, we further investigate the doubly nonlocal problem with singular nonlinearity. Here first, we established a very weak comparison principle we help us to find a relation between the solution of the problem and the solution of a translated problem. By using this relation, we further prove the Hölder regularity and asymptotic behavior of solutions. Moreover, we prove the optimal Sobolev regularity of the solutions. Next, to prove the existence and multiplicity of solutions, we use the nonsmooth analysis of critical point theory.

सार

थीसिस का मुख्य विषय गैर-स्थानीय अंडाकार आंशिक अवकल समीकरणों का अध्ययन है। हाल के समय में, गैर-स्थानीय आंशिक अंतर समीकरणों में रुचि ने इंजीनियरिंग, जीव विज्ञान, भौतिकी और इसके बाद की कई शाखाओं में इसके पर्याप्त मात्रा में अनुप्रयोगों के कारण एक बड़ी वृद्धि प्राप्त की है| गैर-स्थानीय ऑपरेटरों को लंबी दूरी की गतिशीलता पर कब्जा करने की विशेषता है। थीसिस को पाँच अध्यायों में विभाजित किया गया है।

पहले अध्याय में हमने एक संक्षिप्त सर्वेक्षण दिया हैं, बाद के अध्यायों और थीसिस के संगठन में उपयोग किए गए पूर्वग्रहों जो हमारे काम के योगदान का सारांश का परिचय देता हैं।

द्वितीय अध्याय में, हमने गैर-स्थानीय ऑपरेटरों के स्पेक्ट्रल गुणों की अनुसंधान किया हैं। हमने वरिएशनल तरीकों का उपयोग करके विभिन्न प्रकार के गैर-स्थानीय ऑपरेटरों की द्वितीय ईजेंवल्यू और फुसिक स्पेक्ट्रम में पहले गैर-अधम वक्र के अस्तित्व की स्थापना की। इसके बाद, हमने "फैबर-क्रैन" और "गैर-स्थानीय हाँग-क्रान-सेज़गो" जैसी असमानता को विकसित किया है जो की पहले और द्वितीय ईजेंवल्यू के लिए प्रखर निचले सीमा विकसित करती है|

तृतीय अध्याय में हमने महत्वपूर्ण क्रिटिकल नॉनलीनियरिटी वाले चोकार्ड समीकरणों के समाधान के अस्तित्व और बहुलता पर डोमेन की ज्यामिति के प्रभाव को प्रस्तुत करते हैं। हमने वैश्विक कॉम्पैक्टनेस परिणाम की स्थापना की है, जो स्वतंत्र हित का है। हम सबसे पहले महत्वपूर्ण चोकार्ड समस्या के लिए कोरन प्रकार के परिणाम को साबित करते हैं। अगला, हम एक गैर-समरूपता समस्या के चार सकारात्मक समाधानों के अस्तित्व को साबित करने के लिए क्रिटिकल पॉइंट थ्योरी , डिग्री थ्योरी एंड वरिएशनल मेथड्स के साधनों को अनुकूलित करते हैं। इसके अलावा, हमने उत्तल समस्या पर भी चर्चा की है और यह साबित किया है कि डोमेन की टोपोलॉजी की ख़ासियत सकारात्मक समाधानों की संख्या पर निम्न परिबंध देती है।

चौथे अध्याय में हमने दोहरे गैर-स्थानीय समस्याओं के वर्ग के लिए समाधानों की नियमितता प्रदर्शित किया हैं। संक्षेप में, हमने एक व्यापक क्रिटिकल चोकार्ड नॉन लीनियरिटी के साथ फ्रैक्शनल लैपेलियन से जुड़ी समस्या का अध्ययन किया है। हमने समान बाउंड प्राप्त करने के लिए एक नॉन-फोकल ब्रेज़िस-काटो तकनीक विकसित किया हैं। हमने सिंगुलर नॉनलीनारिटी के मामले में भी ब्रेज़िस-काटो तकनीक को दोहरे गैर-स्थानीय ढांचे में विकसित करके कमजोर समाधानों की एकरूपता को

प्राथमिकता दी। हम समस्या के लिए सोबोलेव की तुलन में भारित होल्डर मिनिमाइज़र परिणाम भी स्थापित करते हैं।

पांचवे अध्याय में, हमने सिंगुलर नॉनलीनारिटी के साथ दोगुनी गैर-स्थानीय समस्या का अनुसंधान किया हैं। यहाँ पहले, हमने एक बहुत ही कमजोर तुलना सिद्धांत की स्थापना की जिससे हम समस्या के समाधान और अनुवादित समस्या के समाधान के बीच एक संबंध खोजने में मदद करते हैं। इस संबंध का उपयोग करके, हम आगे समाधान की होल्डर नियमितता और स्पर्शोन्मुख व्यवहार को साबित करते हैं। इसके अलावा, हम समाधानों के सर्वोत्तम सोबोलोव नियमितता को साबित करते हैं। अगला, समाधानों के अस्तित्व और बहुलता को साबित करने के लिए, हम महत्वपूर्ण क्रिटिकल पॉइंट थ्योरी की नॉनस्मूथ विश्लेषण का उपयोग करते हैं।

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List of Symbols

Symbol	Meaning
$\|A\|$	Lebesgue measure of a set $A \subset \mathbb{R}^N$.
$\mathbf{B}_r(x)$	Ball of radius r centered at x in $\mathbb{R}^N$.
$C_c^\infty(\Omega)$	Set of infinitely differentiable functions
	with compact support in Ω.
$\\|f\\|_{L^p}$	Norm of f in $L^p(\Omega)$.
u^+	$\max(u,0)$.
u^-	$\max(-u,0)$.
$\mathbb{R}^N_+$	$\{x \in \mathbb{R}^N \mid x_N > 0\}$.
$*$	the standard convolution operator.
$H^{-1}(\Omega)$	Dual space of $H_0^1(\Omega)$.
$\\| \cdot \\|_*$	Norm in dual space.
$H_0^1(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.
$\\|u\\|_{H_0^1}$	$\left(\int_\Omega \|\nabla u\|^2 dx\right)^{\frac{1}{2}}$, norm of u in $H_0^1(\Omega)$.
$\\|u\\|_{W_0^{1,p}}$	$\left(\int_\Omega \|\nabla u\|^p dx\right)^{\frac{1}{p}}$, norm of u in $W_0^{1,p}(\Omega)$.
Δ	Laplacian.
Δ_p	p-Laplacian.
$\\|u\\|_{L^\infty}$	Norm of u in L^∞ space.
$B_\rho^X(u)$ and $\bar{B}_\rho^X(u)$	Open and closed ball centered at u with radius ρ in X space.
$B_\rho^d(u)$ and $\bar{B}_\rho^d(u))$	Open and closed ball centered at u with radius ρ in $C_d^0(\overline{\Omega})$ space .
$\\|u\\|_{W^{\alpha,p}(\Omega)}$	Norm of u in the fractional order spaces $W^{\alpha,p}(\Omega)$.
$[u]_{H^s(A)}$	$\int_A \int_A \frac{(u(x)-u(y))^2}{\|x-y\|^{N+2s}}\ dxdy$.