

Improved bounds for the max-flow min-multicut ratio for planar and $K_{r,r}$ -free graphs

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Abstract

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We consider the version of the multicommodity flow problem in which the objective is to maximize the sum of commodities routed. Garg, Vazirani and Yannakakis proved that the minimum multicut and maximum flow ratio for this problem can be bounded by $O(\log k)$, where k is the number of commodities. In this note we improve this ratio to $O(1)$ for planar graphs, and more generally to $O(r^3)$ for graphs with an excluded $K_{r,r}$ minor. The proof is based on the network decomposition theorem of Klein, Plotkin and Rao. Our proof is constructive and yields approximation algorithms, with the same factors, for the minimum multicut problem on such networks.

Keywords: Analysis of algorithms; computational complexity; graph theory

1. Introduction

The multicommodity flow problem involves simultaneously shipping several different commodities from their respective sources to their sinks in a single network so that the total amount of flow

going through each edge does not exceed its capacity. We consider the version of the problem where there are no specified demands, instead we wish to maximize the sum of the amounts shipped in all commodities.

The notion of a *multicut* generalizes that of a cut, and is defined as a set of edges whose removal disconnects each source from its corresponding sink. Clearly, maximum multicommodity flow is bounded by the minimum capacity of a multicut. Equality between the maximum flow and the minimum multicut has been established in some special cases, e.g. if there are only two commodities [4]. However, equality does not hold

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in general; consider, for example, a star on 4 nodes with unit edge capacities, and 3 commodities connecting the three pairs of degree one nodes – max flow is $3/2$, whereas minimum multicut is 2.

Leighton and Rao [7] first proved bounds on the minimum-cut maximum-flow ratio for multicommodity flow problems. They considered a different formulation of the multicommodity flow problem in which a demand, d_i , is specified for each commodity i , and the objective is to maximize value z such that zd_i demand can be satisfied simultaneously for every commodity $i = 1, \dots, k$. The corresponding minimization problem is to find a cut that minimizes the ratio of capacity of cut and demand across the cut. Leighton and Rao considered the special case of this problem with one unit of demand between each pair of nodes, and proved a tight $O(\log n)$ bound (modulo constant factors) on the ratio. Klein, Agrawal, Ravi and Rao [5] extended this to arbitrary demands, and proved a bound of $O(\log C \log D)$, where C is the sum of capacities of all edges in the graph, and D is the sum of all demands; both are assumed to be integral. The works of Tragoudas [10], Garg, Vazirani and Yannakakis [3], and Plotkin and Tardos [9] improved this bound to $O(\log^2 k)$, and got over the restriction that capacities and demands be integral. Klein, Plotkin, and Rao [6] gave a decomposition theorem for graphs that do not have a $K_{r,r}$ minor, and used this decomposition to further improve these min-cut max-flow bounds for such graphs to $O(1)$ for the uniform demands case and $O(\log k)$ for the arbitrary demands case.

Garg, Vazirani and Yannakakis [3] have used the techniques developed for the multicommodity flow problem with demands to prove a tight $O(\log k)$ bound (modulo constant factors) for the min-multicut and max-flow ratio. In this note we combine the decomposition theorem of Klein, Plotkin and Rao [6] and the techniques of Garg, Vazirani and Yannakakis [3] to improve the bound to $O(r^3)$ for networks that do not contain a $K_{r,r}$ minor. We also give an efficient $O(r^3)$ factor approximation algorithm for the problem of finding a minimum multicut in such networks. This problem was shown NP-hard for planar graphs in

[1]. In [2], it was shown NP-hard and MAX SNP-hard for trees; as a consequence of the latter fact, the problem does not have a polynomial time approximation scheme (assuming $P \neq NP$). For trees, the bound on min-multicut max-flow ratio can be further improved to 2 [2].

2. Preliminaries

An instance of the *multicommodity flow problem* consists of an undirected graph $G = (V, E)$, a non-negative capacity $u(vw)$ for every edge $vw \in E$, and k source-sink pairs $s_i, t_i \in V$ for $i = 1, \dots, k$ referred to as commodities. We will denote the number of nodes by n , and the number of edges by m .

A flow f_i of commodity i in G from node s_i to node t_i is defined as a collection of paths from s_i to t_i , with associated real values. Let \mathcal{P}_i denote a collection of paths from s_i to t_i in G , and let $f_i(P)$ be a nonnegative value for every P in \mathcal{P}_i . The value of the flow thus defined is $\sum_{P \in \mathcal{P}_i} f_i(P)$, which is the total flow delivered from s_i to t_i . The amount of flow of commodity i through an edge (vw) is given by

$$f_i(vw) = \sum \{f_i(P) : P \in \mathcal{P}_i \text{ and } (vw) \in P\}.$$

A *feasible multicommodity flow* f in G consists of a flow f_i from s_i to t_i for each commodity $1 \leq i \leq k$. We require that $f(vw) \leq u(vw)$ for every edge $vw \in E$, where we use $f(vw) = \sum_{i=1}^k f_i(vw)$ to denote the total amount of flow on the edge vw . The value of the multicommodity flow thus defined is the sum of the values of the flows of each commodity. We will refer to the problem of maximizing the value of a feasible multicommodity flow as the *maximum multicommodity flow problem*.

A *multicut* F is a subset of the edges $F \subseteq E$ such that deleting F separates all source-sink pairs. A *minimum multicut* is a multicut with minimum total capacity. Because of the capacity constraints, the capacity of a multicut is clearly an upper bound on the maximum flow value.

The maximum multicommodity flow problem, as defined above, is a linear program in an expo-

nential number of nonnegative variables: one for each path connecting a source–sink pair. The only constraints are the m capacity constraints $f(vw) \leq u(vw)$. The linear program can be rewritten to be of polynomial size; introduce nonnegative variables $x_i(vw)$ and $x_i(wv)$ to indicate the amount of commodity i for $i = 1, \dots, k$ going through edge $vw \in E$ from v to w , and from w to v , respectively. The flow variables used above can be expressed as $f_i(vw) = x_i(vw) + x_i(wv)$. The polynomial-sized linear program corresponding to the maximum flow problem consists of kn equations describing the flow conservation constraints for every commodity at every node, and m capacity constraints.

The linear programming dual of the maximum multicommodity flow problem is equivalent to the following. There is a nonnegative variable $l(vw)$ corresponding to each edge. The constraints require that the length, $\text{dist}_l(s_i, t_i)$, of the shortest path subject to the length function l connecting each source–sink pair must be at least 1, for $i = 1, \dots, k$. The objective is to minimize the total sum $\sum_{(vw)} l(vw)u(vw)$. Linear programming duality states that this minimum is equal to the maximum flow value. The best way to understand this bound is to think of the network as a pipe system, with edge (vw) corresponding to a pipe of length $l(vw)$ and cross-section $u(vw)$. The sum $\sum_{(vw)} l(vw)u(vw)$, minimized by the dual program, is the total volume of the pipe system. The constraint that the shortest path connecting each source–sink pair must be at least 1 guarantees that every unit of flow routed must use at least one unit of volume in the pipes. Therefore, the total volume of the system is an obvious upper bound on the maximum flow value.

3. The main theorem

In this section we prove the main theorem. The key fact used by us is the following decomposition theorem due to Klein, Plotkin and Rao [6]; it improves on previous decomposition theorems in case the given graph does not contain a $K_{r,r}$ minor. A subset of nodes $S \subseteq V$ has *weak diameter* α if every pair of nodes in S is at distance at

most α in the original graph (and not necessarily in the subgraph induced on S , as in the case of *strong diameter*).

Theorem 3.1 (Klein, Plotkin and Rao [6]). *Given a graph with capacities u on its edges and parameters δ and r one can find, in polynomial time, either a $K_{r,r}$ minor in the graph or an edge separator of total capacity $O(rU/\delta)$ whose removal yields components of weak diameter at most $O(r^2\delta)$, where U is the sum of all capacities.*

Theorem 3.2. *Given an undirected network $G = (V, E)$ with capacities $u(vw)$ on its edges and k source–sink pairs, such that G does not contain a $K_{r,r}$ minor, the ratio of minimum multicut and maximum flow is at most $O(r^3)$. Moreover, we can find, in polynomial time, a multicut in such a graph having capacity within an $O(r^3)$ factor of the max-flow value (and hence also the capacity of the optimal multicut).*

Proof. The high level idea is to modify G into a graph G' such that the edge separator found by the decomposition theorem in G' yields a multicut in G . Graph G' is obtained by subdividing edges of G ; this ensures that G' will also not have a $K_{r,r}$ minor. Each edge of G' is assigned the same capacity as the edge it corresponds to in G . Moreover, the subdivision is done in such a way that each source–sink pair is $\Omega(r^2\delta)$ apart, for a suitably chosen δ . Then, the edge separator found by the decomposition theorem of [6] will be a multicut in G' , and so the corresponding edges will also be a multicut in G of at most the same capacity. The capacity of the edge separator is proportional to the total capacity of all edges in G' . To minimize the latter, we first obtain an optimal solution to the dual LP – this gives the most cost effective way of assigning lengths to edges, so that each source–sink pair is at least distance one apart.

Let l be an optimal solution to the dual LP, and let $\alpha = \sum_{(vw)} l(vw)u(vw)$. By LP duality, α is also the maximum flow value. Now, an edge (vw) of length $l(vw)$ is subdivided into $[l(vw)s]$ edges, where s will be fixed later. Then, the length of the shortest path connecting a source–sink pair,

$\text{dist}_T(s_i, t_i) \geq s$ for every $i = 1, \dots, k$. Let us choose δ so that after the decomposition, the weak diameter of the resulting components is less than s , i.e., $O(r^2\delta) < s$. Clearly, choosing $\delta = s/O(r^2)$ is sufficient.

Now the capacity of edges in the new graph is

$$\sum_{(vw) \in E} \tilde{I}(vw)u(vw) \leq s\alpha + U.$$

Hence, the capacity of the separator obtained is $O(r^3(s\alpha + U)/s)$. By setting $s\alpha = U$, i.e. $s = U/\alpha$, the resulting multicut in G has capacity $O(r^3\alpha)$.

Now consider the running time of the resulting algorithm. The decomposition algorithm of Klein, Plotkin and Rao [6] consists of r breadth-first searches on the subdivided graph. Note that the subdivided graph can have a pseudopolynomial number of nodes, but a breadth-first search can be implemented without explicitly subdividing the graph, and therefore the r breadth-first searches can be implemented in $O(rm)$ time. The most time-consuming part of the algorithm is solving the dual linear program corresponding to the multicommodity flow problem. Notice that an approximately optimal dual solution would suffice. The properties of the optimal dual solution that we need is that each source-sink pair must be at least distance 1 apart (dual feasibility), and the total volume, α , of the system should be at most a small constant multiple of the maximum flow value (approximate optimality).

Approximately optimal solutions to the maximum multicommodity flow problem can be obtained using the framework of Plotkin, Shmoys and Tardos [8]. For simplicity of stating running times we shall ignore polylogarithmic factors. We say that $f(n)$ is $O^*(g(n))$ if $f(n) = O(g(n) \log^c n)$ for some constant c . The approximation algorithm of Plotkin, Shmoys and Tardos [8] for the maximum multicommodity flow problem consist of $O^*(m)$ iterations, where in each iteration one needs to find the shortest paths between corresponding source-sink pairs. Such an iteration can

be implemented in $O^*(km)$ time, so the resulting running time for obtaining an approximately optimal solution is $O^*(km^2)$. \square

Corollary 3.3. *The ratio of minimum multicut and maximum flow in planar graphs is at most $O(1)$, and there is a constant factor approximation algorithm for the problem of finding a minimum multicut in such graphs.*

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